Logic and Proof Techniques

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1 The language of mathematics

Real analysis is concerned not only with calculating quantities but also with explaining why mathematical assertions are true. A calculation may suggest a pattern, but a proof establishes that the pattern must hold in every case covered by the assertion. Before we can construct proofs, we must understand the logical language in which mathematical assertions are expressed. This section begins with statements and logical connectives, then introduces the quantifiers that govern variables. These ideas will provide the grammar for the proof techniques developed later.

1.1 Statements and logical connectives

Definition 1.1. A *statement* is a declarative sentence that has exactly one truth value: true or false. We need not know its truth value in order for it to be a statement. A sentence containing an unassigned variable, such as $x^2 - 5x + 6 = 0$, is an *open sentence*; it becomes a statement after the variable is assigned or quantified.

Let p and q be statements. The fundamental connectives are:

Negation: $\neg p$, read "not p."

Conjunction: $p \wedge q$, read "p and q."

Disjunction: $p \vee q$, read "p or q." In mathematics, *or* is inclusive: one or both alternatives may hold.

Implication: $p \Rightarrow q$, read "if p, then q." Here p is the hypothesis (or antecedent) and q is the conclusion (or consequent).

Equivalence: $p \Leftrightarrow q$, read "p if and only if q." It abbreviates $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

The implication $p \Rightarrow q$ is false only when p is true and q is false. An implication makes no claim about what happens when its hypothesis is false. This convention is essential for universally quantified theorems.

Table 1: Truth table for the fundamental logical connectives.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

The same implication can appear in several forms:

if p, then q; q if p; p only if q; p is sufficient for q; q is necessary for p.

Example 1.2. The sentence "A function is differentiable at c only if it is continuous at c" has hypothesis "the function is differentiable at c" and conclusion "the function is continuous at c." Thus it means

$$\text{differentiable at } c \Rightarrow \text{continuous at } c.$$

The word *only* does not reverse the arrow.

1.2 Logical equivalence and negation

A *tautology* is a compound statement that is true for every assignment of truth values. Two statements are *logically equivalent* when they have the same truth value in every case. The following equivalences are used constantly in analysis:

$$\begin{aligned}
\neg(p \wedge q) &\Leftrightarrow (\neg p) \vee (\neg q), \\
\neg(p \vee q) &\Leftrightarrow (\neg p) \wedge (\neg q), \\
\neg(p \Rightarrow q) &\Leftrightarrow p \wedge \neg q, \\
p \Rightarrow q &\Leftrightarrow \neg q \Rightarrow \neg p.
\end{aligned}$$

The first two are De Morgan's laws. The third says precisely what it takes to refute an implication. The fourth justifies proof by contrapositive.

Example 1.3 (Necessary and sufficient conditions)**.** The statement "A sequence can converge only if it is bounded" means that convergence implies boundedness. Here convergence is sufficient for boundedness, whereas boundedness is necessary for convergence. The converse is false: the sequence $a_n = (-1)^n$ is bounded but does not converge.

Similarly, "continuity at c is necessary for differentiability at c" means that differentiability implies continuity. Again the converse fails. The function $f(x) = |x|$ is continuous at 0, but it is not differentiable there. These examples show why the words *necessary*, *sufficient*, and *only if* must be read carefully.

1.3 Quantifiers and domains

If $P(x)$ is an open sentence, then

$$\forall x \in S,\ P(x) \qquad \text{and} \qquad \exists x \in S \text{ such that } P(x)$$

are statements. The first is universal; the second is existential. The set S is the *domain of discourse*. A quantified statement is incomplete or ambiguous unless that domain is understood.

Example 1.4. The statement $\forall x,\ x^2 \geq x$ is true when the domain is $\mathbb{Z}$, but false when the domain is $\mathbb{R}$, since $x = 1/2$ is a counterexample.

To negate a quantified statement, switch the quantifier and negate the property:

$$\neg\Big(\forall x \in S,\ P(x)\Big) \Leftrightarrow \exists x \in S \text{ such that } \neg P(x),$$

$$\neg\Big(\exists x \in S \text{ such that } P(x)\Big) \Leftrightarrow \forall x \in S,\ \neg P(x).$$

In words, "not every" means "at least one fails," whereas "none" means "every one fails."

Example 1.5 (Negating an analysis statement)**.** Consider the statement

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}, \quad |x - 1| < \delta \Rightarrow |(2x+3) - 5| < \varepsilon.$$

Move from left to right, switching each quantifier, and negate the final implication. The negation is

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in \mathbb{R} \quad \text{such that} \quad |x - 1| < \delta\ \wedge\ |(2x+3) - 5| \geq \varepsilon.$$

Notice that δ may depend on ε, but it may not depend on x. Quantifier order records these dependencies.

Example 1.6 (The order of quantifiers)**.** The statement

$$\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} \text{ such that } y > x$$

is true: after x is given, we may choose $y = x + 1$. By contrast,

$$\exists y \in \mathbb{R}\ \forall x \in \mathbb{R}, \quad y > x$$

is false because it asserts that one fixed real number is greater than every real number. Its negation is

$$\forall y \in \mathbb{R}\ \exists x \in \mathbb{R} \text{ such that } y \leq x.$$

Thus changing the order of quantifiers changes which quantities may depend on which others, and can change the truth value of the statement.

2 Techniques of Proof

Testing examples is *inductive reasoning*: it can suggest a conjecture, but it does not prove a universal claim. A single counterexample, however, does disprove such a claim.

Example 2.1. For several positive integers, $n^2 + n + 17$ is prime. This evidence does not prove that it is always prime. At $n = 16$,

$$16^2 + 16 + 17 = 289 = 17^2,$$

so the universal claim is false.

Deductive reasoning starts from definitions, axioms, previously proved results, and valid logical steps. It produces a conclusion that must follow from the hypotheses.

2.1 The logical bridge and direct proof

Most theorems have the form $p \Rightarrow q$. A direct proof assumes p and constructs a logical bridge to q. The building blocks are:

1. definitions;
2. accepted axioms or basic properties;
3. previously proved theorems; and
4. statements logically implied by earlier steps.

When planning, work forward from the hypothesis and backward from the conclusion. When writing, present a forward argument in which every symbol is introduced and every important implication is justified.

Proposition 2.2 (A model direct proof)**.** *For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x \in \mathbb{R}$,*

$$|x - 1| < \delta \quad \Longrightarrow \quad |(2x + 3) - 5| < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Choose $\delta = \varepsilon/2$, which is positive. Let $x \in \mathbb{R}$ and suppose that $|x - 1| < \delta$. Then

$$|(2x + 3) - 5| = |2x - 2| = 2|x - 1| < 2\delta = \varepsilon.$$

Therefore the required δ exists for every $\varepsilon > 0$. $\square$

The discovery process runs backward: the desired inequality will hold if $2|x-1| < \varepsilon$, suggesting $\delta = \varepsilon/2$. The written proof runs forward and verifies that the choice works.

Proposition 2.3. *If m and n are odd integers, then $m+n$ is even.*

Proof. Since m and n are odd, there are integers j and k such that $m = 2j+1$ and $n = 2k+1$. Consequently,

$$m+n = (2j+1)+(2k+1) = 2(j+k+1).$$

The number $j+k+1$ is an integer, so $m+n$ is twice an integer. Hence $m+n$ is even. $\square$

This proof is short because the definitions do most of the work. Notice that the proof uses different integer witnesses j and k; the definition of odd does not require two odd integers to have the same witness.

2.2 Indirect proofs

A direct proof is often the most natural approach, but it is not always the most efficient one. Logical equivalences allow us to replace an implication by another statement whose hypotheses are easier to use. The most important such replacement is the contrapositive. A second indirect method assumes that the desired conclusion fails and shows that this assumption is impossible. Some arguments instead divide the domain into an exhaustive collection of cases.

2.2.1 Contrapositive, converse, and inverse

Associated with $p \Rightarrow q$ are three related implications:

Table 2: Statements associated with an implication.

Name	**Form**	**Relationship**
Original	$p \Rightarrow q$	Equivalent to the contrapositive
Contrapositive	$\neg q \Rightarrow \neg p$	Equivalent to the original
Converse	$q \Rightarrow p$	Not generally equivalent to the original
Inverse	$\neg p \Rightarrow \neg q$	Equivalent to the converse

Proposition 2.4 (Proof by contrapositive)**.** *Let $m \in \mathbb{Z}$. If m^2 is even, then m is even.*

Proof. We prove the contrapositive. Suppose that m is odd. Then $m = 2k+1$ for some $k \in \mathbb{Z}$. Hence

$$m^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1.$$

Since $2k^2+2k \in \mathbb{Z}$, the integer m^2 is odd. Therefore, if m^2 is even, then m is even. $\square$

Contrapositive is especially useful when the conclusion is negative, when the negation of the conclusion gives concrete algebraic information, or when the hypothesis is difficult to use directly.

Remark 2.5 (A common logical error). Proving $q \Rightarrow p$ does not prove $p \Rightarrow q$. For example, $x > 2 \Rightarrow x^2 > 4$ is true, but its converse is false over $\mathbb{R}$: $x = -3$ satisfies $x^2 > 4$ but not $x > 2$.

2.2.2 Proof by contradiction

To prove a statement p, assume $\neg p$ and derive a contradiction. To prove $p \Rightarrow q$, assume both p and $\neg q$, then derive a known falsehood. The contradiction must arise from the entire set of assumptions, not from an algebra mistake.

Proposition 2.6. *There is no greatest real number.*

Proof. Suppose, for contradiction, that there is a greatest real number M. Since $M + 1 \in \mathbb{R}$ and $M + 1 > M$, the number M is not greatest. This is a contradiction. Therefore no greatest real number exists. $\square$

Proposition 2.7. *If $x \in \mathbb{R}$ and $x > 0$, then $1/x > 0$.*

Proof. Suppose that $x > 0$ but $1/x \leq 0$. Multiplication by the positive number x preserves the inequality, so

$$1 = x(1/x) \leq x \cdot 0 = 0,$$

contradicting $1 > 0$. Thus $1/x > 0$. $\square$

2.3 Proof by cases

If the hypothesis can be exhausted by cases $p_1, \ldots, p_k$, prove the conclusion in every case. The cases need not be mutually exclusive, but they must cover every possibility.

Proposition 2.8. *For every $x \in \mathbb{R}$, $x \leq |x|$.*

Proof. Let $x \in \mathbb{R}$. If $x \geq 0$, then $|x| = x$, so $x \leq |x|$. If $x < 0$, then $x < 0 < -x = |x|$. The two cases exhaust all real numbers, so the result follows. $\square$

Proposition 2.9. *For every integer n, the integer $n(n+1)$ is even.*

Proof. Let $n \in \mathbb{Z}$. If n is even, then $n(n+1)$ is even because it has the even factor n. If n is odd, then $n+1$ is even, so $n(n+1)$ is again even. Every integer is either even or odd; therefore the conclusion holds for every $n \in \mathbb{Z}$. □

The proof could be compressed to the observation that one of two consecutive integers is even. The case-based version makes the logical structure explicit: the cases are exhaustive, and the same conclusion is obtained in each case.

2.4 Universal and existential proofs

To prove $\forall x \in S,\ P(x)$, begin with an *arbitrary* $x \in S$. Use only properties shared by every element of S. To prove $\exists x \in S$ such that $P(x)$, produce a witness x, verify that $x \in S$, and verify $P(x)$.

Example 2.10 (Existence and uniqueness)**.** There exists a unique real number x such that $2x + 3 = 7$.

For existence, take $x = 2$, and check that $2(2) + 3 = 7$. For uniqueness, suppose $y \in \mathbb{R}$ also satisfies $2y + 3 = 7$. Then $2y = 4$, hence $y = 2$. Thus the witness exists and no other witness works.

An existence proof must not define the witness using an object whose existence has not been established. Likewise, an arbitrary element is not the same as a convenient example.

2.5 Choosing a proof method

There is no algorithm that always identifies the best proof method. The form of the statement and the available definitions nevertheless provide useful clues. When the hypotheses translate immediately into equations or inequalities, a direct proof is usually appropriate. A contrapositive proof is attractive when the conclusion is negative or when its negation provides concrete information. A contradiction is natural when a theorem rules out the existence of an object, or when the hypotheses and the negated conclusion cannot coexist. A piecewise definition or a natural division of the domain often suggests proof by cases. Finally, an existential conclusion invites the construction of a witness.

These observations are guidelines rather than rules. The same theorem may have several correct proofs, and the shortest proof is not always the most illuminating. In analysis, the most productive first step is often to expand the definitions in the hypothesis and conclusion and then ask what information is available at each end of the logical bridge.

2.6 A common error: proving the converse

Suppose one wishes to prove: "If m^2 is odd, then m is odd." The argument

> Suppose m is odd. Then $m = 2k + 1$, and consequently m^2 is odd.

does not prove the stated theorem. It proves the converse: if m is odd, then m^2 is odd. In this particular situation the converse happens to be true, but that does not establish the original implication. A correct proof of the original statement can use the contrapositive: if m is even, then $m = 2k$ for some integer k, and hence $m^2 = 2(2k^2)$ is even.

This error is avoided by writing down the exact hypothesis and conclusion before beginning a proof. The opening assumption must match either the original hypothesis, the negation of the conclusion in a contrapositive proof, or the hypothesis together with the negation of the conclusion in a contradiction proof.

2.7 Standards for proof writing

A proof is a piece of mathematical writing, not merely a sequence of symbolic calculations. Variables should be introduced together with their domains, and displayed equations should be connected by complete sentences that explain why each step follows. When the proof is indirect, the method should be announced so that the reader understands the role of the opening assumption. Definitions and previously established results should be cited at the point where they do the essential work.

Quantifier dependencies deserve particular attention. If a theorem begins "for every $\varepsilon > 0$, there exists $\delta > 0$," then the chosen δ may depend on ε, but it must be fixed before the later variable x is introduced. A proof that chooses δ in terms of x does not respect the order of the quantifiers and therefore proves a different statement.

The process of discovering a proof is usually less orderly than the finished argument. Scratch work may proceed backward from the conclusion, test examples, or contain incomplete chains of implications. The final proof reorganizes those ideas into a coherent forward explanation. It begins from assumptions permitted by the chosen method and ends by explicitly connecting the argument to the claimed conclusion.