Real Analysis

Thomas R. Cameron

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1 Key Topics

In this review, we cover the pertinent definitions, theorems, and proof techniques for Exam 1. In particular, we review basic proof techniques, sets, relations, functions, cardinality, ordered fields, the completeness axiom, the real numbers, and compact sets.

2 Basic Proof Techniques

Let p and q be statements. In the implication $p \Rightarrow q$, p is the hypothesis and q is the conclusion. If $p \Rightarrow q$ is true, then a direct proof would verify this by assuming the hypothesis and constructing a sequence of true statements until the conclusion is reached. A contrapositive proof makes use of the following logical equivalence

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)\,. \tag{1}$$

A proof by contradiction uses the following logical equivalence

$$(\neg p \Rightarrow c) \Leftrightarrow p, \tag{2}$$

where c is a contradiction, i.e., a statement that is always false. To prove the implication $p \Rightarrow q$ using proof by contradiction, we make use of the following logical equivalence

$$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)\,. \tag{3}$$

Hence, to prove $p \Rightarrow q$ by contradiction, we assume p and $\neg q$ and show that this implies a contradiction.

Let $P(n)$ be a statement for all n. The principal of mathematical induction states that $P(n)$ is true for all $n \in \mathbb{N}$ if the following conditions hold:

a. $P(1)$ is true, and

b. for all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

We showed that the principal of mathematical induction is equivalent to the well-ordering principal of $\mathbb{N}$, which we state below.

Axiom 2.1 (Well-Ordering Principle of $\mathbb{N}$)**.** Every non-empty subset $S \in \mathbb{N}$ has an element $m \in S$ such that $m \leq k$ for all $k \in S$.

3 Sets

A set is a collection of elements, which satisfy the 9 set axioms, see Wikipedia reference link. We don't have the time to cover these axioms in full detail but the *axiom of choice* will show up again.

Given two sets A and B, the Cartesian product of A and B is defined by

$$A \times B = \{(a,b) \colon\ a \in A\ \wedge\ b \in B\}\,.$$

A relation between A and B is any subset $R \subseteq A \times B$. We say that $R \subseteq A \times A$ is an equivalence relation if for all $x, y, z \in A$ the following conditions hold:

a. $(x,x) \in R$

b. $(x,y) \in R \;\Rightarrow\; (y,x) \in R$

c. $((x,y) \in R \;\wedge\; (y,z) \in R) \;\Rightarrow\; (x,z) \in R$

We say that $f \subseteq A \times B$ is a function if

$$((a,b) \in f \;\wedge\; (a,b') \in f) \;\Rightarrow\; b = b'.$$

The domain and range of f, respectively, are defined as follows

$$\begin{aligned} \operatorname{dom}(f) &= \{a \in A\colon\; \exists b \in B \;\ni\; (a,b) \in f\}\,, \\ \operatorname{rng}(f) &= \{b \in B\colon\; \exists a \in A \;\ni\; (a,b) \in f\}\,. \end{aligned}$$

We write $f\colon A \to B$ if $\operatorname{dom}(f) = A$. Furthermore, we say that f is surjective if $\operatorname{rng}(f) = B$ and we say that f is injective if

$$((a,b) \in f \;\wedge\; (a',b) \in f) \;\Rightarrow\; a = a'.$$

We say that f is bijective if it is both surjective and injective. Given a relation $f \subseteq A \times B$, we define its inverse relation as follows

$$f^{-1} = \{(b,a) \in B \times A\colon\; (a,b) \in f\}\,. \tag{4}$$

We say that a set S is countable if S is finite or there exists a bijection from S to $\mathbb{N}$, i.e., S is countable if there exists a bijection $f\colon\; S \to \{1,2,\ldots,n\}$, for some $n \in \mathbb{N}$, or if there exists a bijection $f\colon\; S \to \mathbb{N}$. If S is not countable, then we say that S is uncountable.

4 The Real Numbers

The real numbers, denoted $\mathbb{R}$, satisfy the ordered field axioms and the completeness axiom, all of which are listed below.

Axiom 4.1 (Field $\mathbb{R}$)**.**

a. $\forall x,y \in \mathbb{R},\; x+y \in \mathbb{R}$

b. $\forall x,y \in \mathbb{R},\; x+y = y+x$

c. $\forall x,y,z \in \mathbb{R},\; x+(y+z) = (x+y)+z$

d. There is a unique $0 \in \mathbb{R}$ such that $x+0 = x,\; \forall x \in \mathbb{R}$

e. $\forall x \in \mathbb{R},\; \exists -x \in \mathbb{R} \;\ni\; x+(-x) = 0$

f. $\forall x,y \in \mathbb{R},\; x \cdot y \in \mathbb{R}$

g. $\forall x,y,z \in \mathbb{R},\; x \cdot (y \cdot z) = (x \cdot y) \cdot z$

h. There is a unique $1 \in \mathbb{R}$ such that $x \cdot 1 = x,\; \forall x \in \mathbb{R}$

i. $\forall x \in \mathbb{R} \setminus \{0\},\; \exists 1/x \in \mathbb{R} \;\ni\; x \cdot (1/x) = 1$

j. $\forall x,y,z \in \mathbb{R},\; x \cdot (y+z) = x \cdot y + x \cdot z$

Axiom 4.2 (Ordering $\mathbb{R}$)**.**

a. For all $x,y \in \mathbb{R}$, exactly one of the relations $x = y$, $x > y$, $x < y$ holds

b. $\forall x,y,z \in \mathbb{R},\; (x < y) \wedge (y < z) \Rightarrow x < z$

c. $\forall x, y, z \in \mathbb{R},\ x < y \Rightarrow x + z < y + z$

d. $\forall x, y, z \in \mathbb{R},\ (x < y) \wedge (z > 0) \Rightarrow xz < yz$

Axiom 4.3 (Completeness of $\mathbb{R}$)**.** Let $S \subseteq \mathbb{R}$ be non-empty. If S is bounded above, then $\sup S$ exists in $\mathbb{R}$.

The fact that $\mathbb{R}$ is an ordered field allows us to define the absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases} \tag{5}$$

Many of the central ideas in analysis are dependent on the notion of two points being "close" to each other, where the distance between two points is given by the absolute value of their difference. A neighborhood of $x \in \mathbb{R}$ is defined by

$$N(x;\epsilon) = \{y \in \mathbb{R}:\ \ |x - y| < \epsilon\},$$

where ϵ is the referred to as the radius of the neighborhood. The deleted neighborhood is defined by

$$N^*(x;\epsilon) = (y \in \mathbb{R}:\ 0 < |y - x| < \epsilon\} = N(x;\epsilon) \setminus \{x\}.$$

Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then x is an interior point of S if

$$\exists \epsilon > 0\ \ni\ N(x;\epsilon) \subseteq S,$$

x is a boundary point of S if

$$\forall \epsilon > 0,\ N(x;\epsilon) \cap S \neq \emptyset\ \wedge\ N \cap (\mathbb{R} \setminus S) \neq \emptyset,$$

and x is an accumulation point of S if

$$\forall \epsilon > 0,\ N^*(x;\epsilon) \cap S \neq \emptyset.$$

The set S is said to be closed if it contains all of its boundary points, or equivalently if it contains all if its accumulation points.

Let $S \subseteq \mathbb{R}$ and let $\mathcal{F}$ be a set of open sets. We say that $\mathcal{F}$ is an open cover of S if

$$S \subseteq \bigcup_{A \in \mathcal{F}} A.$$

If there exists a finite set of open sets $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ is also n open cover of S, we say that S is compact. Equivalently, by the Heine-Borel theorem, S is compact if and only if S is closed and bounded.

5 Exercises

I. Create a truth table to establish the logical equivalences in (1)–(3).

II. Prove by induction that $7^n - 4^n$ is a multiple of 3, for all $n \in \mathbb{N}$.

III. Let A and B be sets, $f \subseteq A \times B$ be a relation, and let f^{-1} be the inverse relation as defined in (4). Prove the following implications:

a. If f is injective, then f^{-1} is a function.

b. If f is a function, then f^{-1} is injective.

c. If $\operatorname{dom}(f) = A$, then f^{-1} is surjective.

IV. Prove that if S is countable and $T \subseteq S$, then T is countable.

V. Let $\mathbb{F}$ denote the set of all rational functions, i.e., all quotients of polynomials:

$$\frac{anx^n + \cdots + a_1x + a_0}{b_kx^k + \cdots + b_1x + b_0},$$

where all coefficients are real numbers and $b_k \neq 0$. Using the well-known rules for adding, subtracting, multiplying, and dividing polynomials, it is clear that $\mathbb{F}$ is a field. Furthermore, we can define an order on $\mathbb{F}$ as follows: A rational function is positive if and only if $a_n \cdot b_k > 0$. Then, given two rational functions p/q and f/g, we say that

$$\frac{p}{q} > \frac{f}{g} \Leftrightarrow \frac{p}{q} - \frac{f}{g} > 0.$$

a. Explain why both $\mathbb{N}$ and $\mathbb{R}$ are subsets of $\mathbb{F}$.

b. Show that $\mathbb{F}$ satisfies the field axioms.

c. Show that $\mathbb{F}$ satisfies the order axioms.

d. Show that $\mathbb{F}$ does not satisfy the Archimedean property, i.e., find a $z \in \mathbb{F}$ such that $z > n$ for all $n \in \mathbb{N}$.

e. Show that $\mathbb{F}$ does not satisfy the completeness axiom, i.e., find a subset $B \subseteq \mathbb{F}$ such that B is bounded above, but B has no least upper bound.

VI. Prove the Bolzano-Weirstrass Theorem: If $S \subseteq \mathbb{R}$ is bounded and contains infinitely many points, then S contains at least one accumulation point in $\mathbb{R}$.