Real Analysis

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1 Key Topics

In this review, we cover the pertinent definitions, theorems, and proof techniques for Exam 1. In particular, we review sequences, limits of functions, continuity, differentiability, and the mean value theorem.

2 Sequences

A sequence is a function $s\colon \mathbb{N} \to \mathbb{R}$, where $s_n = s(n)$ denotes the nth element of the sequence. We say that the sequence s converges to $L \in \mathbb{R}$ if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{R}\ \ni\ n > N \Rightarrow |s_n - L| < \epsilon. \tag{1}$$

If s converges to L, then we write $\lim_{n\to\infty} s_n = L$. In this case, we reference L as the limiting value of the sequence s. If s does not converge, then we say it diverges.

Below are some basic properties of convergent sequences.

Proposition 2.1. *Suppose that the sequence s converges. Then, the follow properties hold:*

I. The range of s is bounded.

II. The limiting value of s is unique.

Below are the algebraic properties of the limits of sequences.

Proposition 2.2. *Let $s\colon \mathbb{N} \to \mathbb{R}$ and $t\colon \mathbb{N} \to \mathbb{R}$ be convergent with limits L and L', respectively. Then,*

I. $\lim\limits_{n\to\infty} (s_n + t_n) = L + L'$

II. $\lim\limits_{n\to\infty} (ks_n) = kL,\ \forall k \in \mathbb{R}$

III. $\lim\limits_{n\to\infty} (s_n \cdot t_n) = L \cdot L'$

IV. $\lim\limits_{n\to\infty} \left(\dfrac{s_n}{t_n}\right) = \dfrac{L}{L'}$

The following result leads to the squeeze theorem for sequences.

Theorem 2.3. *Let $s\colon \mathbb{N} \to \mathbb{R}$ and $t\colon \mathbb{N} \to \mathbb{R}$ be convergent with limits L and L', respectively. If $s_n \le t_n$ for all $n \in \mathbb{N}$, then $L \le L'$.*

The following result is often refereed to as the ratio test.

Theorem 2.4. *Let $s\colon \mathbb{N} \to \mathbb{R}$ with $s_n > 0$ for all $n \in \mathbb{N}$. If*

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = L < 1,$$

then $\lim_{n\to\infty} s_n = 0$.

2.1 Monotone Sequences

A sequence $s\colon \mathbb{N} \to \mathbb{R}$ is monotone if it is increasing ($s_n \leq s_{n+1},\ \forall n \in \mathbb{N}$) or if it is decreasing ($s_n \geq s_{n+1},\ \forall n \in \mathbb{N}$).

Recall that all convergent sequences are bounded. The following result states that, for monotone sequences, being convergent and being bounded are equivalent properties.

Theorem 2.5. *A monotone sequence converges if and only if it is bounded.*

2.2 Cauchy Sequences

A sequence $s\colon \mathbb{N} \to \mathbb{R}$ is Cauchy if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{R} \ \ni \ n, m > N \Rightarrow |s_n - s_m| < \epsilon.$$

For real sequences, the property of being convergent and the property of being Cauchy is equivalent. We establish that fact through the following results.

Lemma 2.6. *Every Cauchy sequence is bounded.*

Proposition 2.7. *Every convergent sequence is Cauchy.*

Proposition 2.8. *Every real Cauchy sequence is convergent.*

2.3 Subsequences

Let $s\colon \mathbb{N} \to \mathbb{R}$ and let $n\colon \mathbb{N} \to \mathbb{N}$ be a strictly increasing sequence ($n_k < n_{k+1},\ \forall k \in \mathbb{N}$).

Below are several interesting results regarding subsequences.

Proposition 2.9. *Let $s\colon \mathbb{N} \to \mathbb{R}$ be convergent with limit L. Then, every subsequence of s converges to L.*

Proposition 2.10. *Let $s\colon \mathbb{N} \to \mathbb{R}$ and let $\operatorname{rng}(s)$ denote the range of s. If $\operatorname{rng}(s)$ has an accumulation point, denoted by a, then there is a subsequence of s that converges to a.*

Proposition 2.11. *Every bounded sequence has a convergent subsequence.*

3 Limits of Functions

Let $S \subseteq \mathbb{R}$, $f\colon S \to \mathbb{R}$, $L \in \mathbb{R}$, and c be any accumulation point of S. Then, we say that f converges to L as x approaches c if

$$\forall \epsilon > 0,\ \exists \delta > 0 \ \ni \ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

If f converges to L as x approaches c, then we write $\lim_{x \to c} f(x) = L$. In this case, we reference L as the limiting value of f as x approaches c. If f does not converge as x approaches c, then we say it diverges.

Note that c is an accumulation point of S if

$$\forall \epsilon > 0,\ N^*(c; \epsilon) \cap S \neq \emptyset.$$

Furthermore, we can rephrase this definition in terms of sequences. In particular, c is an accumulation point of S if and only if

$$\exists s\colon \mathbb{N} \to S \ \ni \ \operatorname{rng}(s) \subseteq S \setminus \{c\},\ \lim_{n \to \infty} s_n = c.$$

Armed with the sequence definition of an accumulation point, we can rephrase the limit definition of a function in terms of sequences.

Theorem 3.1. *Let $S \subseteq \mathbb{R}$, $f\colon S \to \mathbb{R}$, $L \in \mathbb{R}$, and c be any accumulation point of S. Then, $\lim_{x \to c} f(x) = L$ if and only if for all $s\colon \mathbb{N} \to \mathbb{R}$ such that $\operatorname{rng}(s) \subseteq S \setminus \{c\}$ and $\lim_{n \to \infty} s_n = c$, we have*

$$\lim_{n \to \infty} f(s_n) = L.$$

Therefore, everything we know holds for sequential limits will also hold for limits of functions. In particular, Proposition 2.2 and Theorem 2.3 hold for limits of functions. We summarize these results below.

Proposition 3.2. *Let $f\colon S \to \mathbb{R}$, $g\colon S \to \mathbb{R}$, and c be an accumulation point of S. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = L'$, then the following hold*

I. $\lim_{x\to c} (f(x) + g(x)) = L + L'$,

II. $\lim_{x\to c} (kf(x)) = kL$, *for all* $k \in \mathbb{R}$,

III. $\lim_{x\to c} (f(x) \cdot g(x)) = L \cdot L'$,

IV. $\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{L'}$, *if* $L' \neq 0$.

Theorem 3.3. *Let $f\colon S \to \mathbb{R}$, $g\colon S \to \mathbb{R}$, and c be an accumulation point of S. Suppose that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = L'$. If $f(x) \leq g(x)$ for all $x \in S \setminus \{c\}$, then $L \leq L'$.*

4 Continuity and Differentiation

Let $S \subseteq \mathbb{R}$, $f\colon S \to \mathbb{R}$, and $c \in \mathbb{S}$. Then, f is continuous at c if

$$\forall \epsilon > 0,\ \exists \delta > 0 \ \ni \ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

If f is continuous at c for all $c \in S$, then we say that f is continuous on S.

We can restate the definition of continuity two different ways. First, the following result breaks continuity into two cases based on whether or not c is an accumulation point.

Theorem 4.1. *Let $S \subseteq \mathbb{R}$, $f\colon S \to \mathbb{R}$, and $c \in \mathbb{S}$. If c is an isolated point of S, the f is continuous at c. If c is an accumulation point of S, then f is continuous at c if and only if*

$$\lim_{x\to c} f(x) = f(c).$$

Second, the following result rephrases the definition of continuity in terms of sequential limits.

Theorem 4.2. *Let $S \subseteq \mathbb{R}$, $f\colon S \to \mathbb{R}$, and $c \in \mathbb{S}$. Then, f is continuous at c if and only if for all $s\colon \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to\infty} s_n = c$, we have*

$$\lim_{n\to\infty} f(s_n) = f(c).$$

Below are several important results regarding continuous functions over compact domains.

Lemma 4.3. *Let $S \subseteq \mathbb{R}$ be compact (closed and bounded). Then, any continuous function $f\colon S \to \mathbb{R}$ is bounded.*

Theorem 4.4. *Let $S \subseteq \mathbb{R}$ be compact (closed and bounded) and let $f\colon S \to \mathbb{R}$ be continuous. Then, $f(S) \subseteq \mathbb{R}$ is compact.*

Corollary 4.5. *Let $S \subseteq \mathbb{R}$ be compact (closed and bounded) and let $f\colon S \to \mathbb{R}$ be continuous. Then, f achieves both its maximum and minimum values on S.*

One stronger type of continuity is uniform continuity. Recall that $f\colon S \to \mathbb{R}$ is uniformly continuous on S if

$$\forall \epsilon > 0,\ \exists \delta > 0 \ \ni \ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Below are several important results regarding uniform continuity.

Proposition 4.6. *Let $f\colon S \to \mathbb{R}$. If f is uniformly continuous on S, then f is continuous on S.*

Theorem 4.7. *Let $f\colon S \to \mathbb{R}$ be continuous. If S is compact, then f is uniformly continuous on S.*

Let $I \subseteq \mathbb{R}$ be an interval and let $f\colon I \to \mathbb{R}$. Then, f is differentiable at $c \in I$ if there exists an $L \in \mathbb{R}$ such that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L.$$

In this case, we say that L is the derivative of f at c and we write $f'(c) = L$. If f is differentiable at all $c \in I$, then we say that f is differentiable on I and we define the derivative function $f'\colon I \to \mathbb{R}$ by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

for all $c \in I$.

Using the sequential criterion for limits of a function, see Theorem 3.1, we can rephrase the derivative definition in terms of sequences.

Theorem 4.8. *Let $I \subseteq \mathbb{R}$ be an interval and let $f\colon I \to \mathbb{R}$. Then, f is differentiable at $c \in I$ if and only if there exists an $L \in \mathbb{R}$ such that for all $s\colon \mathbb{N} \to I$, where $\operatorname{rng}(s) \subseteq I \setminus \{c\}$ and $\lim_{n\to\infty} s_n = c$, we have*

$$\lim_{n \to \infty} \frac{f(s_n) - f(c)}{s_n - c} = L.$$

We can use the sequential limit criterion for derivatives to show that differentiability implies continuity.

Theorem 4.9. *Let $I \subseteq \mathbb{R}$ be an interval, $f\colon I \to \mathbb{R}$, and $c \in I$ If f is differentiable at c, then f is continuous at c.*

Below are the basic properties of differentiation.

Proposition 4.10. *Let I be an interval and $c \in I$. Suppose that $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ are differentiable at c. Then, the following properties hold*

I. $(f+g)'(c) = f'(c) + g'(c)$

II. $(kf)'(c) = kf'(c), \ \forall k \in \mathbb{R}$

III. $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

IV. $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}$

In addition, we have the power rule which states that

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all $n \in \mathbb{R}$ (we proved this for all natural numbers n). Finally, there is the chain rule which states that

$$(f \circ g)'(c) = f'(g(c))g'(c).$$

Note that we did not prove the chain rule in this class.

5 Mean Value Theorem

The mean value theorem states that under suitable conditions there always a point at which the instantaneous rate of change of a function is equal to its average rate of change. To prove the mean value theorem, we require Lemma 5.1. Recall that c is a relative max of f if there exists a $\delta > 0$ such that

$$\forall x \in N(c;\delta),\ f(x) \leq f(c).$$

Also, c is a relative min of f if there exists a $\delta > 0$ such that

$$\forall x \in N(c;\delta),\ f(x) \geq f(c).$$

If c is either a relative min or a relative max, we say that c is a relative extrema of f.

Lemma 5.1. *Suppose that $f\colon (a,b) \to \mathbb{R}$ is differentiable. If $c \in (a,b)$ is a relative extrema, then $f'(c) = 0$.*

Next, we use Lemma 5.1 to prove Rolle's theorem.

Theorem 5.2 (Rolle's)**.** *Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there is a $c \in (a,b)$ such that $f'(c) = 0$.*

Finally, we use Rolle's theorem to prove the mean value theorem.

Theorem 5.3. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then, there exists a $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Below are several results that can be proved using the mean value theorem.

Proposition 5.4.

Let I be an open interval and let $f\colon I \to \mathbb{R}$ be differentiable. Then,

I. *If $f'(x) \geq 0$ for all $x \in I$, then f is monotone increasing.*

II. *If $f'(x) \leq 0$ for all $x \in I$, then f is monotone decreasing.*

Theorem 5.5. *Let f and its first n derivatives be continuous on $[a,b]$ and differentiable on (a,b) and let $x_0 \in [a,b]$. Then, for each $x \in [a,b]$, with $x \neq x_0$, there exists a c between x and x_0 such that*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

6 Exercises

I. Prove Proposition 2.1

II. Prove Proposition 2.2

III. Prove Proposition 2.7

IV. Prove Proposition 2.10

V. Prove Theorem 3.1

VI. Prove Lemma 4.3

VII. Prove Proposition 4.10

VIII. Prove Proposition 5.4

IX. Prove Theorem 5.5

X. Show that $s_n = (-1)^n$ does not converge by showing that it is not Cauchy.

XI. Show that $s_n = 1 - 1/n$ does converge by showing that it is bounded and monotone increasing. What does this sequence converge to?

XII. Use Theorem 3.1 and the squeeze theorem to show that $\lim_{x\to 0} x\cos(1/x) = 0$.

XIII. Use Theorem 3.1 to show that $\lim_{x\to 0} \cos(1/x)$ does not exist.

XIV. Show that $f(x) = x^2$ is continuous on $\mathbb{R}$ but is not uniformly continuous.

XV. Show that $f(x) = |x|$ is continuous at $x = 0$ but is not differentiable.

XVI. Use the mean value theorem on the interval $[36, 40]$ to estimate the value of $\sqrt{40}$.