Riemann Integrals

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1 Darboux Integrals

Let $f\colon [a,b] \to \mathbb{R}$ be bounded. Then, there exists $m, M \in \mathbb{R}$ such that

$$m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a),$$

for any partition P of $[a,b]$. Therefore, the upper and lower Darboux sums for f form a bounded set, which guarantees the existence of the upper and lower Darboux integrals. In particular, the upper Darboux integral is defined by

$$U(f) = \inf\{U(f,P)\colon P \text{ is a partition of } [a,b]\}$$

and the lower Darboux integral is defined by

$$L(f) = \sup\{L(f,P)\colon P \text{ is a partition of } [a,b]\}$$

The following lemma shows that the lower Darboux integral is always bounded above by the upper Darboux integral.

Lemma 1.1. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded. Then, $L(f) \leq U(f)$.*

Proof. Define $A = \{L(f,P)\colon P \text{ is a partition of } [a,b]\}$ and $B = \{U(f,P)\colon P \text{ is a partition of } [a,b]\}$. For the sake of contradiction, suppose that $\sup(A) > \inf(B)$. Define $\epsilon = (\sup(A) - \inf(B))/2$. Then,

$$\inf(B) + \epsilon = \frac{\sup(A) + \inf(B)}{2} < \frac{\sup(A) + \sup(A)}{2} = \sup(A)$$

Since $\epsilon > 0$, $\inf(B) + \epsilon$ is not a lower bound on B. Therefore, there exists a partition Q of $[a,b]$ such that

$$\inf(B) \leq U(f,Q) < \inf(B) + \epsilon < \sup(A).$$

Since $\inf(B) + \epsilon$ is not a upper bound on A, there exists a partition P of $[a,b]$ such that

$$\inf(B) \leq U(f,Q) < \inf(B) + \epsilon < L(f,P) \leq \sup(A).$$

However, this implies that

$$U(f,Q) < L(f,P),$$

which contradicts Corollary 2.2 (Darboux Sums Notes). Therefore, $\sup(A) \leq \inf(B)$. $\square$

2 Riemann Integrals

If $L(f) = U(f)$, then we say that f is *Riemann integrable*. In this case, we denote the Riemann integral by

$$\int_a^b f(x)dx = L(f) = U(f).$$

The following theorem gives a necessary and sufficient condition for when a bounded function is Riemann integrable.

Theorem 2.1. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded. Then, f is Riemann integrable if and only if for all $\epsilon \in \mathbb{R}_{>0}$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.*

Proof. Suppose that f is Riemann integrable. Let $\epsilon \in \mathbb{R}_{>0}$. Then, there exists a partition P of $[a,b]$ such that

$$L(f,P) > L(f) - \frac{\epsilon}{2}.$$

Similarly, there exists a partition Q of $[a,b]$ such that

$$U(f,Q) < U(f) + \frac{\epsilon}{2}.$$

Since $P \cup Q$ is refinement of both P and Q, Theorem 2.1 (Darboux Sums Notes) implies that

$$L(f) - \frac{\epsilon}{2} < L(f,P) \leq L(f,P \cup Q) \leq U(f,P \cup Q) \leq U(f,Q) < U(f) + \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned} U(f,P \cup Q) - L(f,P \cup Q) &< \Big(U(f) + \frac{\epsilon}{2}\Big) - \Big(L(f) - \frac{\epsilon}{2}\Big) \\ &= (U(f) - L(f)) + \epsilon = \epsilon. \end{aligned}$$

Conversely, suppose that for all $\epsilon \in \mathbb{R}_{>0}$ there exists a partition P of $[a,b]$ such that $U(f,P) < L(f,P) + \epsilon$. Then,

$$U(f) \leq U(f,P) < L(f,P) + \epsilon \leq L(f) + \epsilon.$$

Therefore, $U(f) \leq L(f)$. By Lemma 1.1, $L(f) \leq U(f)$. Thus, $L(f) = U(f)$ and it follows that f is Riemann integrable. $\square$

As an example, define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then, for any partition P of $[0,1]$, $L(f,P) = 0$ and $U(f,P) = 1$. Therefore, Theorem 2.1 states that f is not Riemann integrable.

As another example, define $f\colon [0,1] \to \mathbb{R}$ by $f(x) = x$. For $n \in \mathbb{N}$, let $P = \{0, 1/n, 2/n, \ldots, 1\}$ be a partition of $[0,1]$. Then,

$$U(f,P) - L(f,P) = \frac{n^2+n}{2n^2} - \frac{n^2-n}{2n^2} = \frac{1}{n}.$$

For each $\epsilon > 0$, there exists a $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Therefore, Theorem 2.1 states that f is Riemann integrable.

As a final example, define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 1 & \text{if } x = 0, \\ 0 & \text{if } x = 1. \end{cases}$$

For $n \in \mathbb{N}$, let $P = \{0, 1/n, 2/n, \ldots, 1\}$ be a partition of $[0,1]$. Then,

$$U(f,P) - L(f,P) = \frac{n^2+n-2}{2n^2} + \frac{1}{n} - \frac{n^2-3n+2}{2n^2} = \frac{6n-4}{2n^2} < \frac{3}{n}.$$

For each $\epsilon > 0$, there exists a $n \in \mathbb{N}$ such that $\frac{3}{n} < \epsilon$. Therefore, Theorem 2.1 states that f is Riemann integrable.