Riemann Integration Worksheet

Math 312: Concepts of Real Analysis

Name: _____ **Date:** _____

Directions. Use your notes on Darboux sums and the Riemann integral. Justify each claim and identify the definitions or theorems you use. Continue on additional paper as needed.

1. **Monotone functions (Theorem 4.2).**

   Let $f : [a, b] \to \mathbb{R}$ be monotone, with $a < b$. We will prove that f is Riemann integrable.

   (a) Explain why the endpoint values bound f. For the uniform partition $P = \{x_0, \ldots, x_n\}$ into n subintervals, give the common width and identify the infimum m_i and supremum M_i of f on $[x_{i-1}, x_i]$ in both the nondecreasing and nonincreasing cases.

   (b) Write $U(f, P) - L(f, P)$ as a sum involving $|f(x_i) - f(x_{i-1})|$. Explain why the differences inside the absolute values have the same weak sign, then use telescoping to express the gap in terms of a, b, n, $f(a)$, and $f(b)$.

   (c) Given $\varepsilon > 0$, choose n so that the gap is less than ε. State why this proves Riemann integrability.

2. An integrable function with two jumps.

Define $f : [0, 3] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } 1 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Identify the discontinuities using one-sided limits. Use three function values to show that f is neither nondecreasing nor nonincreasing.

(b) Let $0 < h < 1/2$ and $P_h = \{0, 1 - h, 1 + h, 2 - h, 2 + h, 3\}$. Find the infimum and supremum of f on each of the five closed subintervals. Use these heights and the subinterval widths to compute $L(f, P_h)$ and $U(f, P_h)$.

(c) Given $\varepsilon > 0$, choose h so that the Darboux gap is less than ε. Use the Darboux criterion to conclude that f is integrable. Then use the bounds from part (b) to determine $\int_0^3 f(x)\,dx$.

3. Changing finitely many values (Theorem 5.1).

Let $a < b$. Suppose $f, g : [a, b] \to \mathbb{R}$ agree except at finitely many points and f is Riemann integrable. We will prove that g is Riemann integrable with the same integral as f.

(a) Explain why there are common bounds $m < M$ for the values of both functions. Handle the case in which f and g agree everywhere. For the remaining parts, let $k \geq 1$ be the number of points where they differ.

(b) Fix $\varepsilon > 0$. Use the Darboux criterion to choose a partition P_0 with $U(f, P_0) - L(f, P_0) < \varepsilon/2$. Form P by adjoining the points of a uniform partition whose widths are less than $\varepsilon/[8k(M - m)]$. Explain why the gap for f stays below $\varepsilon/2$. Count how many closed subintervals can contain a single changed point, including when it is a partition point. Use this count to show that the total width w of the subintervals containing changed points is less than $\varepsilon/[4(M - m)]$.

(c) Compare the infima and suprema of f and g on subintervals with and without changed points. Use your comparison to bound each of $|L(g, P) - L(f, P)|$ and $|U(g, P) - U(f, P)|$ by $(M - m)w$.

(d) Use part (c) to show that the interval $[L(f, P) - (M - m)w,\ U(f, P) + (M - m)w]$ contains all four sums for P and has length less than ε. Apply the Darboux criterion to g. Then explain why the two integrals lie in this interval and why they must be equal.