Sequences

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MATH 312

1 Sequences and Convergence

1.1 The Definition of Convergence

A *sequence of real numbers* is a function $s\colon \mathbb{N} \to \mathbb{R}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$. We write $s_n = s(n)$ and denote the sequence by $(s_n)_{n=1}^{\infty}$, or simply (s_n). Its index records the position of a term. Unlike a set, a sequence retains order and repetition: the sequences $((-1)^n)$ and $((-1)^{n+1})$ have the same range $\{-1, 1\}$ but different terms at every index.

For instance, $(1/n) = (1, 1/2, 1/3, \ldots)$ and

$$\left(\frac{n}{2n+1}\right) = \left(\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots\right).$$

We graph a sequence using the separate points (n, s_n). The sequence is defined only at natural-number indices, so we do not connect its points.

Definition 1.1. A sequence (s_n) *converges to* $L \in \mathbb{R}$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |s_n - L| < \epsilon.$$

We write $\lim_{n\to\infty} s_n = L$. A sequence *diverges* if it does not converge to any real number.

The inequality means $s_n \in (L - \epsilon, L + \epsilon)$. Thus every neighborhood of L contains *all sufficiently late terms*, not merely some terms. The order of the quantifiers matters: after ϵ is given, we choose one N, which must work for every $n \geq N$. This N may depend on ϵ.

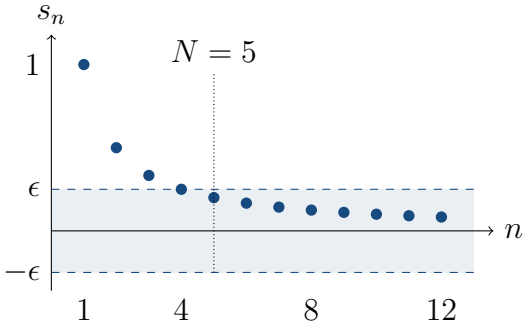

Figure 1. For $s_n = 1/n$ and $\epsilon = 1/4$, the choice $N = 5$ works. The term $s_4 = 1/4$ lies on the boundary, so it does not satisfy the strict inequality.

Connection with topology. Every neighborhood of L contains all terms with $n \geq N$ for some $N \in \mathbb{N}$. If $\lim_{n\to\infty} s_n = L$ and every s_n lies in a closed set $F \subseteq \mathbb{R}$, then $L \in F$. Otherwise $\mathbb{R} \setminus F$ is an open set containing L, so some neighborhood of L is disjoint from F. Convergence would place all sufficiently late terms in that neighborhood, a contradiction.

1.2 Proving Convergence

A calculation can suggest a limit; the definition verifies it. We first estimate the error $|s_n - L|$, then choose an index that makes that error smaller than an arbitrary $\epsilon > 0$. The Archimedean property guarantees natural numbers larger than any prescribed real bound.

Example 1.1 (Reciprocals). Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ with $N > 1/\epsilon$. Then, for every $n \geq N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

We need a valid N, not necessarily the smallest one.

Example 1.2 (A rational expression). Prove that $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$. The error simplifies to

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \frac{1}{4n+2} < \frac{1}{4n}.$$

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ with $N > 1/(4\epsilon)$. For $n \geq N$, the error is less than $1/(4n) \leq 1/(4N) < \epsilon$.

Example 1.3 (Oscillation with shrinking size). The sequence $s_n = (-1)^n/n$ alternates in sign. Nevertheless,

$$|s_n - 0| = \frac{1}{n},$$

so the same choice of N as in Example 1.1 proves $\lim_{n\to\infty} s_n = 0$. The terms need not approach their limit monotonically.

Example 1.4 (An error involving a square root). Let $s_n = \sqrt{n^2+n} - n$. Rationalizing suggests the limit $1/2$:

$$s_n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+1/n}+1}.$$

Set $a_n = \sqrt{1+1/n} \geq 1$. Rationalizing once more gives

$$\left|s_n - \frac{1}{2}\right| = \frac{a_n - 1}{2(a_n+1)} = \frac{1}{2n(a_n+1)^2} \leq \frac{1}{8n}.$$

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ with $N > 1/(8\epsilon)$. For $n \geq N$, the error is less than ϵ, proving $\lim_{n\to\infty} s_n = 1/2$.

Example 1.5 (Geometric decay). If $0 < r < 1$, then $\lim_{n\to\infty} r^n = 0$. Write $1/r = 1 + h$ with $h > 0$. Bernoulli's inequality $(1+h)^n \geq 1 + nh$ follows by induction: it holds for $n = 1$, and $(1+nh)(1+h) \geq 1 + (n+1)h$. Thus

$$0 < r^n \leq \frac{1}{1+nh} < \frac{1}{nh}.$$

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ with $N > 1/(h\epsilon)$. For $n \geq N$, the bound is less than ϵ. If $-1 < r < 0$, apply the same estimate to $|r|^n = |r^n|$; the case $r = 0$ is immediate. Thus $\lim_{n\to\infty} r^n = 0$ whenever $|r| < 1$.

2 Basic Properties and Divergence

2.1 Uniqueness and Boundedness

Proposition 2.1. *A convergent sequence has exactly one limit.*

Proof. Suppose $\lim_{n\to\infty} s_n = L_1$ and $\lim_{n\to\infty} s_n = L_2$, with $L_1 \neq L_2$. Set $\epsilon = |L_1 - L_2|/3 > 0$. Choose N_1, N_2 so that $n \geq N_i$ implies $|s_n - L_i| < \epsilon$, for $i = 1, 2$. For $n \geq \max\{N_1, N_2\}$, the triangle inequality gives

$$|L_1 - L_2| \leq |L_1 - s_n| + |s_n - L_2| < 2\epsilon = \frac{2}{3}|L_1 - L_2|,$$

which is impossible. Thus $L_1 = L_2$. □

Definition 2.2. A sequence (s_n) is *bounded* if there exists $M \geq 0$ such that $|s_n| \leq M$ for every $n \in \mathbb{N}$.

Proposition 2.3. *Every convergent sequence is bounded.*

Proof. Suppose $\lim_{n\to\infty} s_n = L$. With $\epsilon = 1$, choose N so that $n \geq N$ implies $|s_n - L| < 1$. Then $|s_n| \leq |s_n - L| + |L| < 1 + |L|$ for $n \geq N$. Define

$$M = \max\Big(\{|L| + 1\} \cup \{|s_n| : 1 \leq n < N\}\Big).$$

This also bounds the finitely many earlier terms, so $|s_n| \leq M$ for all n. The definition of M works even when $N = 1$. □

2.2 Divergent Sequences

The converse is false: bounded terms need not settle near one value.

Example 2.1 (Persistent oscillation). The sequence $s_n = (-1)^n$ is bounded by 1 but diverges. If it converged to L, use $\epsilon = 1$ and choose N. There are both even and odd indices at least N, so $|1 - L| < 1$ and $|-1 - L| < 1$. But then $2 \leq |1 - L| + |L + 1| < 2$, a contradiction.

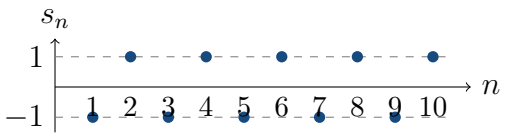

Figure 2. Bounded oscillation does not imply convergence.

Changing finitely many terms does not affect convergence or its limit. The unbounded sequence $s_n = n$ diverges by Proposition 2.3.

We write $\lim_{n\to\infty} s_n = +\infty$ if, for every $A \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n > A$. For $\lim_{n\to\infty} s_n = -\infty$, replace $s_n > A$ by $s_n < A$. These describe divergence,

since infinity is not a real number. For example, $\lim_{n\to\infty} n = +\infty$: given A, choose $N \in \mathbb{N}$ with $N > A$.

Unboundedness alone does not imply divergence to either infinity: $(-1)^n n$ has arbitrarily large positive and negative terms.

3 Limit Theorems

3.1 Algebraic Limit Rules and Order Preservation

The key is to express the new error using the errors $|s_n - L|$ and $|t_n - K|$. When several bounds are needed, taking the maximum of their starting indices makes them hold simultaneously.

Theorem 3.1. *Suppose* $\lim_{n\to\infty} s_n = L$, $\lim_{n\to\infty} t_n = K$, *and* $c \in \mathbb{R}$. *Then*

(a) $\lim_{n\to\infty} (s_n + t_n) = L + K$;

(b) $\lim_{n\to\infty} (cs_n) = cL$;

(c) $\lim_{n\to\infty} (s_n t_n) = LK$;

(d) $\lim_{n\to\infty} \dfrac{s_n}{t_n} = \dfrac{L}{K}$, *provided* $K \neq 0$.

If $K \neq 0$, *there is* $N_0 \in \mathbb{N}$ *such that* $t_n \neq 0$ *for every* $n \geq N_0$. *Any undefined early quotient terms may be assigned arbitrary values without affecting the limit.*

Proof. (a) Let $\epsilon > 0$. Choose N_1, N_2 so that $|s_n - L| < \epsilon/2$ for $n \geq N_1$ and $|t_n - K| < \epsilon/2$ for $n \geq N_2$. For $n \geq \max\{N_1, N_2\}$,

$$|(s_n + t_n) - (L + K)| \leq |s_n - L| + |t_n - K| < \epsilon.$$

(b) Let $\epsilon > 0$. If $c = 0$, the sequence is identically zero. Otherwise choose N so that $|s_n - L| < \epsilon/|c|$ for $n \geq N$. Then $|cs_n - cL| = |c||s_n - L| < \epsilon$.

(c) By boundedness, choose $B \geq \max\{1, |K|\}$ with $|s_n| \leq B$ for every $n \in \mathbb{N}$. Add and subtract $s_n K$:

$$|s_n t_n - LK| \leq |s_n||t_n - K| + |K||s_n - L|.$$

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ so that both $|t_n - K|$ and $|s_n - L|$ are less than $\epsilon/(2B)$ for every $n \geq N$. Then

$$|s_n t_n - LK| \leq B\Big(|t_n - K| + |s_n - L|\Big) < \epsilon.$$

(d) Since $\lim_{n\to\infty} t_n = K \neq 0$, choose N_0 so that $|t_n - K| < |K|/2$ for $n \geq N_0$. Then $|t_n| \geq |K| - |t_n - K| > |K|/2 > 0$, so the late denominators are nonzero and bounded away from zero. For these indices,

$$\left|\frac{1}{t_n} - \frac{1}{K}\right| = \frac{|t_n - K|}{|t_n||K|} \leq \frac{2}{|K|^2}|t_n - K|.$$

Given $\epsilon > 0$, choose $N \geq N_0$ so that $|t_n - K| < \epsilon|K|^2/2$ for $n \geq N$. The bound proves $\lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{K}$; the product rule now gives $\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{L}{K}$.

□

Example 3.1 (Using the algebraic rules). Divide numerator and denominator by n^2:

$$\lim_{n\to\infty} \frac{3n^2 - n + 4}{2n^2 + 5} = \lim_{n\to\infty} \frac{3 - 1/n + 4/n^2}{2 + 5/n^2} = \frac{3}{2}.$$

Here $\lim_{n\to\infty} \frac{1}{n} = 0$, products give $\lim_{n\to\infty}(1/n)^2 = 0$, and the denominator tends to the nonzero number 2.

Example 3.2 (Why the denominator condition matters). If $s_n = t_n = 1/n$, both limits are zero, but $s_n/t_n = 1$. If instead $s_n = 1/n$ and $t_n = 1/n^2$, their quotient is n, which diverges. A denominator limit of zero does not determine the quotient's behavior.

Proposition 3.2 (Order preservation). *Suppose* $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$. *If there exists* $N_0 \in \mathbb{N}$ *such that* $a_n \leq b_n$ *for every* $n \geq N_0$, *then* $A \leq B$.

Proof. If $A > B$, set $\epsilon = (A-B)/3$. Choose $N_1, N_2 \in \mathbb{N}$ so that $|a_n - A| < \epsilon$ for $n \geq N_1$ and $|b_n - B| < \epsilon$ for $n \geq N_2$. For $n \geq \max\{N_0, N_1, N_2\}$, $a_n > A - \epsilon > B + \epsilon > b_n$, contradicting $a_n \leq b_n$. □

Strict inequalities may become equalities in the limit: $0 < 1/n$ for every n, but both sides tend to zero.

3.2 The Squeeze Theorem

The algebraic rules require limits for the factors being combined. Comparison can work even when one factor has no limit: instead of following every oscillation, we bound its size. The squeeze theorem requires two bounds with the *same* limit; it does not require the middle sequence to be monotone. Only sufficiently large indices matter.

Theorem 3.3 (Squeeze theorem). *Suppose there exists* $N_0 \in \mathbb{N}$ *such that* $a_n \leq s_n \leq b_n$ *for every* $n \geq N_0$. *If* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, *then* $\lim_{n\to\infty} s_n = L$.

Proof. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ with $N \geq N_0$ such that both $|a_n - L| < \epsilon$ and $|b_n - L| < \epsilon$ for every $n \geq N$. Then

$$L - \epsilon < a_n \leq s_n \leq b_n < L + \epsilon,$$

so $|s_n - L| < \epsilon$. □

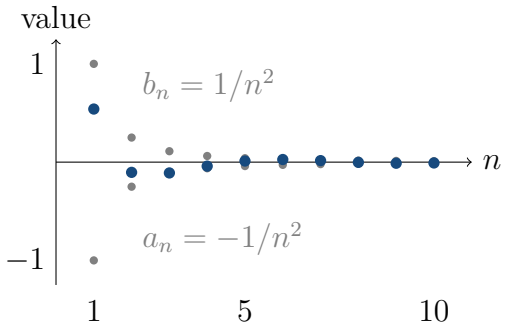

Figure 3. The blue terms $s_n = \cos n/n^2$ are trapped between two bounds tending to zero.

Example 3.3 (Oscillation with a shrinking envelope). Since $-1 \le \cos n \le 1$ and $n^2 > 0$,

$$-\frac{1}{n^2} \le \frac{\cos n}{n^2} \le \frac{1}{n^2}.$$

Both bounds have limit zero, so

$$\lim_{n\to\infty} \frac{\cos n}{n^2} = 0.$$

This adapts the calculus example $x^2 \cos(1/x)$ by setting $x = 1/n$. No limit for the oscillating factor is needed.

Proposition 3.4 (A bounded factor times a sequence converging to zero). *If* (u_n) *is bounded and* $\lim_{n\to\infty} v_n = 0$*, then* $\lim_{n\to\infty}(u_n v_n) = 0$.

Proof. Choose $B \ge 0$ with $|u_n| \le B$. Then $-B|v_n| \le u_n v_n \le B|v_n|$. Since $\lim_{n\to\infty}|v_n| = 0$ directly from the definition, both bounds tend to zero. Apply the squeeze theorem. □

The zero limit is essential: a bounded factor need not converge, so a product with a factor tending to a nonzero number need not converge. For example, $u_n = (-1)^n$ and $v_n = 1$ give a divergent product.

Example 3.4 (A nonzero limit from trigonometric bounds). We can adapt the geometric squeeze for $\sin x/x$ to sequences without using derivatives or assuming trigonometric continuity. All angles are measured in radians. In the unit circle, the inscribed triangle, sector, and tangent triangle have areas $\sin x/2$, $x/2$, and $\tan x/2$. Their inclusion gives $\sin x \le x \le \tan x$ for $0 < x < \pi/2$. Dividing by positive quantities gives

$$\cos x \le \frac{\sin x}{x} \le 1 \quad (0 < x < \pi/2), \qquad |\sin y| \le |y| \quad (|y| < \pi/2).$$

The second inequality follows from $\sin y \le y$ for positive y and oddness for negative y. Using $1 - \cos x = 2\sin^2(x/2)$, it implies $0 \le 1 - \cos x \le x^2/2$. Set $x = 1/n$ to obtain

$$1 - \frac{1}{2n^2} \le \cos(1/n) \le n\sin(1/n) \le 1.$$

The outer sequences both tend to 1, so $\lim_{n\to\infty}\big(n\sin(1/n)\big) = 1$. The geometric inequalities supply the bounds; the sequence squeeze theorem supplies the convergence.