

Subsequences and Function Limits Quiz

Math 312: Concepts of Real Analysis

Name: _____

Date: _____

Directions. You have 20 minutes. State definitions and results precisely, including all hypotheses and quantifiers. Prove the limits requested in Problems 2(c) and 3(b). No external resources or electronic devices are permitted.

1. Subsequences and compactness.

Let (s_n) be a real sequence and $K \subseteq \mathbb{R}$.

- (a) Define a subsequence (s_{n_k}) of (s_n) . State the condition on the selected indices.
- (b) Define a subsequential limit of (s_n) .
- (c) State the Bolzano–Weierstrass theorem for real sequences. State the sequential characterization of compactness. Specify where the subsequence limit must lie.

2. Definitions of function limits.

Let $S \subseteq \mathbb{R}$, $f : S \rightarrow \mathbb{R}$, $c \in S'$, and $L \in \mathbb{R}$.

- (a) State the ϵ - δ definition of $\lim_{x \rightarrow c} f(x) = L$, including the restrictions on x .
- (b) Negate the definition in part (a): state precisely what it means that f does not have this specified limit L at c .
- (c) Use the ϵ - δ definition to prove

$$\lim_{x \rightarrow 2} (x^2 + 1) = 5.$$

3. The sequential criterion and limit laws.

Let $f, g : S \rightarrow \mathbb{R}$ and $c \in S'$.

- (a) State the sequential criterion for $\lim_{x \rightarrow c} f(x) = L$. Include every restriction on the input sequences.
- (b) Let $h : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be defined by

$$h(x) = \begin{cases} x + 1, & x < 0, \\ x + 3, & x > 0. \end{cases}$$

Prove that $\lim_{x \rightarrow 0} h(x)$ does not exist by finding two sequences of nonzero inputs tending to 0 whose image sequences have different limits.

- (c) Suppose $\lim_{x \rightarrow c} f(x) = A$ and $\lim_{x \rightarrow c} g(x) = B$. State all four algebraic limit laws: sum, scalar multiple, product, and quotient. Include the additional hypothesis for the quotient and let the scalar be any $a \in \mathbb{R}$.