

Subsequences and Function Limits Quiz

Math 312: Concepts of Real Analysis

Answer Key

1. Subsequences and compactness.

Let (s_n) be a real sequence and $K \subseteq \mathbb{R}$.

- (a) Define a subsequence (s_{n_k}) of (s_n) . State the condition on the selected indices.
- (b) Define a subsequential limit of (s_n) .
- (c) State the Bolzano–Weierstrass theorem for real sequences. State the sequential characterization of compactness. Specify where the subsequence limit must lie.

Solution.

- (a) A subsequence is $(s_{n_k})_{k=1}^{\infty}$, where $n_k \in \mathbb{N}$ and $n_{k+1} > n_k$ for every $k \in \mathbb{N}$. The selected values themselves need not increase.
- (b) A real number L is a subsequential limit if there exists a subsequence (s_{n_k}) such that $\lim_{k \rightarrow \infty} s_{n_k} = L$.
- (c) Every bounded sequence of real numbers has a subsequence converging to a real number. The set K is compact if and only if every sequence of points of K has a subsequence converging to a point of K .

2. Definitions of function limits.

Let $S \subseteq \mathbb{R}$, $f : S \rightarrow \mathbb{R}$, $c \in S'$, and $L \in \mathbb{R}$.

- (a) State the ϵ - δ definition of $\lim_{x \rightarrow c} f(x) = L$, including the restrictions on x .
- (b) Negate the definition in part (a): state precisely what it means that f does not have this specified limit L at c .
- (c) Use the ϵ - δ definition to prove

$$\lim_{x \rightarrow 2} (x^2 + 1) = 5.$$

Solution.

- (a) For every $\epsilon > 0$, there exists $\delta > 0$ such that, for every $x \in S$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.
- (b) There exists $\epsilon_0 > 0$ such that, for every $\delta > 0$, there exists $x \in S$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon_0$. This rules out the specified value L , but does not rule out every other real limiting value.
- (c) Given $\epsilon > 0$, choose $\delta = \min\{1, \epsilon/5\}$. If $0 < |x - 2| < \delta$, then $|x + 2| \leq |x - 2| + 4 < 5$, and hence

$$|(x^2 + 1) - 5| = |x - 2||x + 2| < 5|x - 2| < 5\delta \leq \epsilon.$$

This proves the limit by the definition.

3. The sequential criterion and limit laws.

Let $f, g : S \rightarrow \mathbb{R}$ and $c \in S'$.

- (a) State the sequential criterion for $\lim_{x \rightarrow c} f(x) = L$. Include every restriction on the input sequences.
- (b) Let $h : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be defined by

$$h(x) = \begin{cases} x + 1, & x < 0, \\ x + 3, & x > 0. \end{cases}$$

Prove that $\lim_{x \rightarrow 0} h(x)$ does not exist by finding two sequences of nonzero inputs tending to 0 whose image sequences have different limits.

- (c) Suppose $\lim_{x \rightarrow c} f(x) = A$ and $\lim_{x \rightarrow c} g(x) = B$. State all four algebraic limit laws: sum, scalar multiple, product, and quotient. Include the additional hypothesis for the quotient and let the scalar be any $a \in \mathbb{R}$.

Solution.

- (a) For $L \in \mathbb{R}$, $\lim_{x \rightarrow c} f(x) = L$ if and only if, for every sequence (x_n) with $x_n \in S \setminus \{c\}$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} x_n = c$, we have $\lim_{n \rightarrow \infty} f(x_n) = L$.
- (b) Take $x_n = -1/n$ and $y_n = 1/n$. Both are nonzero and tend to 0, but

$$\lim_{n \rightarrow \infty} h(x_n) = \lim_{n \rightarrow \infty} (1 - 1/n) = 1, \quad \lim_{n \rightarrow \infty} h(y_n) = \lim_{n \rightarrow \infty} (3 + 1/n) = 3.$$

If a real function limit existed, the sequential criterion would force both image sequences to converge to that limit, contradicting uniqueness of sequence limits. Thus no real limit exists.

- (c) For every $a \in \mathbb{R}$,

$$\begin{aligned} \lim_{x \rightarrow c} (f(x) + g(x)) &= A + B, & \lim_{x \rightarrow c} af(x) &= aA, \\ \lim_{x \rightarrow c} f(x)g(x) &= AB, & \lim_{x \rightarrow c} \frac{f(x)}{g(x)} &= \frac{A}{B} \quad (B \neq 0). \end{aligned}$$

The quotient is defined on $T = \{x \in S : g(x) \neq 0\}$. When $B \neq 0$, this set contains all points of S sufficiently close to c and distinct from c , and $c \in T'$.