Subsequences

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MATH 312

1 Selecting Terms in Order

A subsequence selects infinitely many terms in their original order. Different selections can reveal different limiting behavior.

Definition 1.1. Let (s_n) be a sequence of real numbers. If (n_k) is a sequence of natural numbers satisfying

$$n_1 < n_2 < n_3 < \cdots,$$

then $(s_{n_k})_{k=1}^{\infty}$ is a *subsequence* of (s_n). In function notation, a subsequence is $s \circ \sigma$, where $\sigma(k) = n_k$ is strictly increasing.

There are two indices: n_k is a position in the original sequence; k is the position in the subsequence. The selected *indices* must increase, but the selected *values* need not increase.

Example 1.1 (Different selections). For $s_n = 1/n$, the even selection is $s_{2k} = 1/(2k)$, the odd selection is $s_{2k-1} = 1/(2k-1)$, and the square-index selection is $s_{k^2} = 1/k^2$.

The list $(1/2, 1, 1/3, 1/4, \ldots)$ is not a subsequence: it selects index 2 before index 1. A finite list is not a subsequence either.

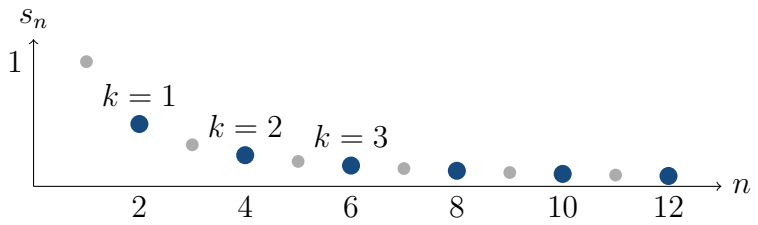

Figure 1. Blue points form (s_{2k}). The horizontal coordinate still records the original index $n = 2k$.

Proposition 1.2. *For every strictly increasing sequence (n_k) in $\mathbb{N}$, we have $n_k \geq k$ for all $k \in \mathbb{N}$.*

Proof. Since $n_1 \geq 1$, the claim holds at $k = 1$. If $n_k \geq k$, then $n_{k+1} \geq n_k + 1 \geq k + 1$, completing the induction. □

Thus selected indices grow beyond every fixed index. Also, a subsequence of a subsequence is a subsequence of the original sequence: if (k_j) increases strictly, then so does (n_{k_j}).

2 Convergence and Divergence Tests

Theorem 2.1. *If* $\lim_{n\to\infty} s_n = L$*, then every subsequence satisfies* $\lim_{k\to\infty} s_{n_k} = L$*.*

Proof. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $|s_n - L| < \epsilon$ for every $n \geq N$. If $k \geq N$, then $n_k \geq k \geq N$ by Proposition 1.2. Hence $|s_{n_k} - L| < \epsilon$, proving the claim. □

The converse holds if *every* subsequence converges to L, since the original sequence is its own subsequence. The existence of just one convergent subsequence is much weaker.

Example 2.1 (Two incompatible limits). Let $s_n = (-1)^n + 1/n$. Then

$$s_{2k} = 1 + \frac{1}{2k}, \qquad s_{2k-1} = -1 + \frac{1}{2k-1}.$$

The limit laws give $\lim_{k\to\infty} s_{2k} = 1$ and $\lim_{k\to\infty} s_{2k-1} = -1$. If (s_n) converged, Theorem 2.1 would force both limits to equal its limit. Thus (s_n) diverges, although it is bounded.

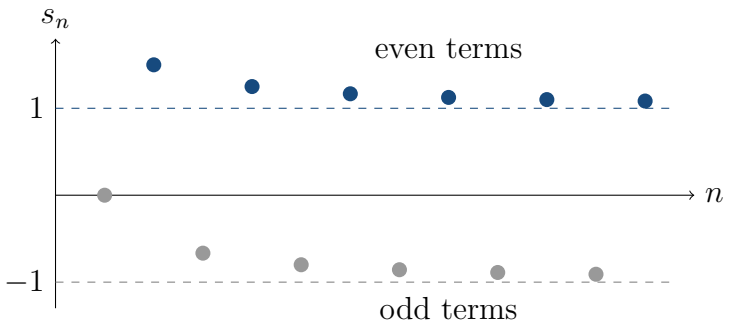

Figure 2. Two subsequences approach different real numbers, ruling out convergence of the full sequence.

Example 2.2 (One convergent subsequence is insufficient). Define $a_{2k} = 0$ and $a_{2k-1} = k$. Its even subsequence converges to 0, while its odd subsequence diverges to $+\infty$. Thus an unbounded sequence can have a convergent subsequence. By contrast, $b_n = n$ has no convergent subsequence: $b_{n_k} = n_k \geq k$, so every subsequence diverges to $+\infty$.

Proposition 2.2 (Even and odd terms). *If* $\lim_{k\to\infty} s_{2k} = L$ *and* $\lim_{k\to\infty} s_{2k-1} = L$*, then* $\lim_{n\to\infty} s_n = L$*.*

Proof. Given $\epsilon > 0$, choose K such that both $|s_{2k} - L| < \epsilon$ and $|s_{2k-1} - L| < \epsilon$ for every $k \geq K$. Every $n \geq 2K$ is $2k$ or $2k-1$ for some $k \geq K$, so $|s_n - L| < \epsilon$. □

For example, if $s_{2k} = 3 + 1/k$ and $s_{2k-1} = 3 - 1/k^2$, then $\lim_{n\to\infty} s_n = 3$. The two selections together cover all positions.

3 Subsequential Limits and Selection

Definition 3.1. A real number L is a *subsequential limit* of (s_n) if some subsequence (s_{n_k}) satisfies $\lim_{k\to\infty} s_{n_k} = L$.

Theorem 3.2 (A selection criterion). *A number $L \in \mathbb{R}$ is a subsequential limit of (s_n) if and only if, for every $\epsilon > 0$ and every $N \in \mathbb{N}$, there exists $n \geq N$ such that $|s_n - L| < \epsilon$.*

Proof. If $\lim_{k\to\infty} s_{n_k} = L$, choose $k \geq N$ sufficiently large that $|s_{n_k} - L| < \epsilon$. Then $n_k \geq k \geq N$.

Conversely, set $n_0 = 0$. After choosing n_{k-1}, apply the stated condition with $N = n_{k-1} + 1$ and $\epsilon = 1/k$ to choose $n_k > n_{k-1}$ with $|s_{n_k} - L| < 1/k$. Since $\lim_{k\to\infty} 1/k = 0$, these estimates prove $\lim_{k\to\infty} s_{n_k} = L$. □

For a subsequential limit, every neighborhood contains terms at arbitrarily late *indices*. For convergence of the full sequence, every neighborhood must contain *all* terms past some index. This distinction explains how oscillating sequences can have several subsequential limits.

Example 3.1 (All subsequential limits). For $s_n = (-1)^n + 1/n$, Example 2.1 supplies the limits -1 and 1. There are no others. Indeed, any convergent subsequence has infinitely many even selected indices or infinitely many odd selected indices. Selecting those positions again gives a further subsequence with limit 1 or -1. Theorem 2.1 says that this further subsequence has the same limit as the convergent subsequence. Thus its limit belongs to $\{-1, 1\}$.

3.1 Connection with Accumulation Points

Let $S = \{s_n : n \in \mathbb{N}\}$ be the range. If L is an accumulation point of S, every neighborhood of L contains infinitely many distinct points of S. To see why, if a neighborhood contained only finitely many points other than L, a smaller neighborhood could exclude all of them, contradicting accumulation.

Only finitely many values occur in any finite initial segment. Consequently every neighborhood of L contains a term at an arbitrarily late index, and Theorem 3.2 gives a subsequence converging to L.

Example 3.2 (Repetition matters). For $s_n = (-1)^n$, the range is the finite set $\{-1, 1\}$, which has no accumulation points. Nevertheless both -1 and 1 are subsequential limits because each value occurs at infinitely many indices. In general, every value repeated infinitely often gives a constant subsequence.

Proposition 3.3 (A Cauchy sequence with one convergent subsequence). *If (s_n) is Cauchy and $\lim_{k\to\infty} s_{n_k} = L$, then $\lim_{n\to\infty} s_n = L$.*

Proof. Given $\epsilon > 0$, choose N such that $|s_n - s_m| < \epsilon/2$ for $m, n \geq N$. Choose k with $n_k \geq N$ and $|s_{n_k} - L| < \epsilon/2$. For every $n \geq N$, $|s_n - L| \leq |s_n - s_{n_k}| + |s_{n_k} - L| < \epsilon$. □

4 Bolzano–Weierstrass for Sequences

Theorem 4.1 (Bolzano–Weierstrass). *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let $S = \{s_n : n \in \mathbb{N}\}$ be bounded. If S is finite, at least one of its values occurs at infinitely many indices; otherwise each value would occur only finitely often, and their finite union would account for only finitely many terms. Selecting those indices gives a constant, hence convergent, subsequence.

If S is infinite, the set form of Bolzano–Weierstrass gives an accumulation point $L \in \mathbb{R}$. The preceding section shows that L is a subsequential limit. More explicitly, set $n_0 = 0$. Each interval $(L-1/k, L+1/k)$ contains infinitely many distinct values of S, whereas the first n_{k-1} terms supply only finitely many values. Thus we can select $n_k > n_{k-1}$ with $|s_{n_k} - L| < 1/k$. This subsequence converges to L. □

The theorem guarantees a convergent selection, not convergence of the full sequence. It is especially useful when a sequence is bounded but no formula suggests a limit. Together with Proposition 3.3, it also gives a short proof of the Cauchy criterion: a real Cauchy sequence is bounded, has a convergent subsequence, and therefore converges.

4.1 Constructing a Convergent Selection

The estimate $|s_{n_k} - L| < 1/k$ controls the error, while $n_k > n_{k-1}$ preserves the order. Both conditions are needed. Merely choosing a point near L for each k could repeatedly select the same position.

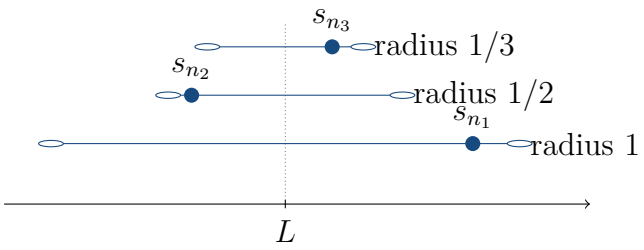

Figure 3. Choose later indices in smaller neighborhoods of L. The selected values need not be monotone. Positions are illustrative.

4.2 Unbounded Sequences

Proposition 4.2. *If (s_n) is unbounded above, it has a strictly increasing subsequence with limit $+\infty$. If it is unbounded below, it has a strictly decreasing subsequence with limit $-\infty$.*

Proof. Every tail of an unbounded-above sequence is unbounded above, since a finite initial segment is bounded. Choose n_1 with $s_{n_1} > 1$. Having selected n_k, choose $n_{k+1} > n_k$ with $s_{n_{k+1}} > \max\{k+1, s_{n_k}\}$. Thus $s_{n_k} > k$, which proves divergence to $+\infty$. Apply this argument to $(-s_n)$ for the second assertion. □

Every unbounded real sequence is unbounded above or below. Its selected infinite limit describes divergence, not convergence to a real number. It may still have other subsequences with finite limits, as in Section 2.

5 Compactness and Detecting Convergence

Theorem 5.1 (Sequential characterization of compactness). *A set $K \subseteq \mathbb{R}$ is compact if and only if every sequence of points of K has a subsequence converging to a point of K.*

Proof. If K is compact, Heine–Borel says it is closed and bounded. A sequence in K is bounded, so Theorem 4.1 supplies a subsequence converging to some $L \in \mathbb{R}$. Closedness forces $L \in K$.

Conversely, assume the stated sequence property. If K were unbounded, choose $x_n \in K$ with $|x_n| > n$. Every subsequence would be unbounded, since $|x_{n_k}| > n_k \geq k$, so none could converge. Hence K is bounded.

If K were not closed, there would be $x \in \mathrm{cl}(K) \setminus K$. For each n, choose $x_n \in K$ with $|x_n - x| < 1/n$. Then $\lim_{n\to\infty} x_n = x$, so every subsequence has limit $x \notin K$, again a contradiction. Thus K is closed and bounded, and Heine–Borel makes it compact. □

Example 5.1 (Why the limit must remain in the set). In $K = (0, 1)$, the sequence $x_n = 1/(n + 1)$ has limit 0. Every subsequence also has limit 0, so none converges to a point of K. The set is bounded but not compact. In contrast, every sequence in $[0, 1]$ has a subsequence with limit in $[0, 1]$.

Theorem 5.2 (A unique subsequential limit). *Let (s_n) be bounded. If every convergent subsequence has the same limit L, then $\lim_{n\to\infty} s_n = L$.*

Proof. Suppose the sequence does not converge to L. Negating convergence gives an $\epsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there is $n \geq N$ with $|s_n - L| \geq \epsilon_0$. Set $n_0 = 0$ and apply this condition with $N = n_{k-1} + 1$ at each step to select $n_k > n_{k-1}$ with $|s_{n_k} - L| \geq \epsilon_0$.

The bounded sequence (s_{n_k}) has a convergent subsequence $(s_{n_{k_j}})$ by Theorem 4.1. It is also a subsequence of (s_n), so its limit is L by assumption. But every selected term stays at least ϵ_0 from L, contradicting convergence to L. □

Boundedness is essential: the sequence $s_{2k} = 0$, $s_{2k-1} = k$ has just one real subsequential limit, 0, but diverges. A convergent subsequence cannot select infinitely many odd indices, since those values are unbounded.

Example 5.2 (Identifying the only possible limit). Suppose $0 \leq a_n \leq 2$ and $\lim_{n\to\infty}(a_n^2 - a_n) = 0$. If $\lim_{k\to\infty} a_{n_k} = L$, the subsequence theorem and limit laws give

$$L^2 - L = \lim_{k\to\infty} (a_{n_k}^2 - a_{n_k}) = 0.$$

Thus every subsequential limit belongs to $\{0, 1\}$. If we also know $a_n \geq 1/2$ for all n, order preservation excludes $L = 0$. Theorem 5.2 then proves $\lim_{n\to\infty} a_n = 1$, without assuming monotonicity.

6 Examples of the Selection Method

6.1 Using a Subsequence to Identify a Limit

Example 6.1 (Roots of a positive number). Fix $0 < x < 1$ and let $a_n = x^{1/n}$. We show that $\lim_{n\to\infty} a_n = 1$ without using logarithms.

First establish convergence. Since $0 < x < 1$, decreasing the exponent increases the value, so $a_{n+1} > a_n$. Also $x \le a_n < 1$. The monotone convergence theorem therefore gives a limit L with $x \le L \le 1$.

Now select the even terms. The identity $a_{2n}^2 = a_n$ and Theorem 2.1 give

$$L^2 = \lim_{n\to\infty} a_{2n}^2 = \lim_{n\to\infty} a_n = L.$$

Hence $L(L-1) = 0$. Because $L \ge x > 0$, we obtain $L = 1$. The subsequence helps identify a limit only after convergence of the full sequence has been established.

6.2 Approaching a Supremum from Within a Set

Example 6.2 (A maximum of a compact set). Let $K \subseteq \mathbb{R}$ be nonempty and compact, and write $M = \sup K$, which exists because K is nonempty and bounded. For every $n \in \mathbb{N}$, the definition of supremum supplies $x_n \in K$ such that

$$M - 1/n < x_n \le M.$$

The squeeze theorem gives $\lim_{n\to\infty} x_n = M$. By the sequential characterization of compactness, some subsequence converges to a point $x \in K$. But Theorem 2.1 says that its limit is M, so uniqueness gives $x = M \in K$. Thus K contains its supremum, which is its maximum. The same construction using the infimum gives a minimum.

This illustrates a method used later in analysis: choose points whose values approach a bound, then use compactness to obtain a limit that remains in the set.

6.3 How the Main Ideas Work Together

Boundedness supplies a convergent subsequence through Bolzano–Weierstrass. Closedness keeps its limit in the set. A Cauchy estimate then turns one convergent subsequence into convergence of the full sequence. Alternatively, if every convergent selection has the same limit, boundedness and Theorem 5.2 force the full sequence to converge.

In a contradiction argument, failure of convergence supplies terms that remain a fixed distance from the proposed limit. Selecting these terms in increasing index order isolates the obstruction. This is the essential step in the proof of Theorem 5.2, and it will recur when we study limits and continuity of functions.

Reading. Steven R. Lay, *Analysis with an Introduction to Proof*, Section 19, on subsequence convergence and existence; see also Sections 14 and 18 for compactness and Cauchy sequences.