

Subsequences Worksheet

Math 312: Concepts of Real Analysis

Answer Key

1. A selection criterion: Theorem 3.2.

Let (s_n) be a real sequence and $L \in \mathbb{R}$. Consider the condition

$$\text{for every } \epsilon > 0 \text{ and every } N \in \mathbb{N}, \quad \text{there exists } n \geq N \text{ with } |s_n - L| < \epsilon. \quad (*)$$

- (a) State the definition of $\lim_{n \rightarrow \infty} s_n = L$. Explain how its quantifiers differ from $(*)$. For $s_n = (-1)^n$ and $L = 1$, verify $(*)$ and explain why the full sequence does not converge to L .
- (b) Suppose $\lim_{k \rightarrow \infty} s_{n_k} = L$. Given $\epsilon > 0$ and $N \in \mathbb{N}$, explain how to choose k so that both $n_k \geq N$ and $|s_{n_k} - L| < \epsilon$. Which property of the selected indices proves $(*)$?
- (c) Conversely, assume $(*)$ and set $n_0 = 0$. At step k , specify the choices of N and ϵ that produce

$$n_k > n_{k-1}, \quad |s_{n_k} - L| < 1/k.$$

Explain why these choices give a subsequence, then prove that its limit is L . Why would choosing a nearby term without requiring a later index be insufficient?

Solution.

- (a) Convergence means that for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that every $n \geq N$ satisfies $|s_n - L| < \epsilon$. Condition $(*)$ requires only one suitable term beyond each prescribed index, equivalently infinitely many suitable indices for each tolerance. For $(-1)^n$, an even $n \geq N$ has error zero from 1. Odd terms have error 2, so convergence to 1 fails, for example with $\epsilon = 1$.
- (b) Choose K such that $|s_{n_k} - L| < \epsilon$ for $k \geq K$, then take $k \geq \max\{K, N\}$. The strictly increasing natural-number indices satisfy $n_k \geq k$, hence $n_k \geq N$. Thus $n = n_k$ works.
- (c) Use $N = n_{k-1} + 1$ and tolerance $1/k$. This yields strictly increasing natural-number indices and error less than $1/k$. Given $\eta > 0$, choose $K > 1/\eta$. For $k \geq K$, $|s_{n_k} - L| < 1/k \leq 1/K < \eta$. A nearby value alone need not occur at a new position: for $s_1 = 0$ and $s_n = 1$ for $n \geq 2$, choosing the first position repeatedly would not produce a subsequence with limit zero.

To construct a subsequence converging to L , control both the indices and the errors.

2. A Cauchy sequence with one convergent subsequence: Proposition 3.3.

Suppose (s_n) is Cauchy and $\lim_{k \rightarrow \infty} s_{n_k} = L$. We will prove that $\lim_{n \rightarrow \infty} s_n = L$ without using the Cauchy convergence criterion.

- (a) State the definition of a Cauchy sequence. Given $\epsilon > 0$, use it to choose $N \in \mathbb{N}$ such that

$$|s_n - s_m| < \epsilon/2 \quad \text{for all } m, n \geq N.$$

Explain why convergence of the subsequence lets you fix one index k with both $n_k \geq N$ and $|s_{n_k} - L| < \epsilon/2$.

- (b) For an arbitrary $n \geq N$, insert the selected term s_{n_k} into an estimate for $|s_n - L|$. Justify each bound and complete the proof. Explain why the same fixed k works for every $n \geq N$ and why closeness of consecutive terms alone would not justify your argument.
- (c) Suppose (a_n) is Cauchy and $a_{3k} = 2 + 1/k$ for every $k \in \mathbb{N}$. Determine the limit of the full sequence and justify your answer. Show that the Cauchy assumption cannot be omitted by defining values at the other indices that make the full sequence diverge.

Solution.

- (a) For every $\eta > 0$ there exists $N \in \mathbb{N}$ such that $|s_n - s_m| < \eta$ for all $m, n \geq N$. Take $\eta = \epsilon/2$. Subsequence convergence supplies K such that $|s_{n_k} - L| < \epsilon/2$ for $k \geq K$. Choose $k \geq \max\{K, N\}$; then $n_k \geq k \geq N$.
- (b) For every $n \geq N$, the triangle inequality gives

$$|s_n - L| \leq |s_n - s_{n_k}| + |s_{n_k} - L| < \epsilon/2 + \epsilon/2 = \epsilon.$$

The Cauchy bound applies because both n and the fixed n_k are at least N . It controls every pair of late indices, so k need not change with n . Consecutive-term bounds do not directly control these possibly distant indices; many small differences may accumulate.

- (c) The subsequence (a_{3k}) has limit 2, so the result just proved gives $\lim_{n \rightarrow \infty} a_n = 2$. Without the Cauchy assumption, set $a_n = 0$ when 3 does not divide n . Then (a_{3k+1}) has limit zero while (a_{3k}) has limit 2, so the sequence diverges.

The triangle inequality combines closeness to a selected term with closeness to the limit.

3. Bolzano–Weierstrass: Theorem 4.1.

Let (s_n) be bounded and put $S = \{s_n : n \in \mathbb{N}\}$. We will prove the sequence form of Bolzano–Weierstrass using its set form.

- (a) Suppose S is finite. Explain why some $L \in S$ occurs at infinitely many indices and how to choose a constant subsequence. For $s_n = (-1)^n$, identify the subsequential limits and explain why neither is an accumulation point of the range. Why does the set form of Bolzano–Weierstrass not apply to this range?
- (b) Suppose S is infinite. State the set form of Bolzano–Weierstrass and apply it to obtain an accumulation point $L \in \mathbb{R}$. State the definition of accumulation point. Explain why every neighborhood of L contains infinitely many distinct points of S : what smaller neighborhood could you choose if there were only finitely many points other than L ?
- (c) Set $n_0 = 0$. Explain why, after selecting n_{k-1} , there is an $n_k > n_{k-1}$ with $|s_{n_k} - L| < 1/k$. Identify precisely where you use infinitely many distinct values rather than merely one nearby value. Complete the proof of Theorem 4.1 and explain why it does not prove convergence of the full sequence.

Solution.

- (a) If every value in finite S occurred only finitely often, there would be only finitely many terms. Some value therefore occurs at infinitely many indices; list those indices in increasing order. The selected values are constant. For $(-1)^n$, the subsequential limits are -1 and 1 , but each is isolated in the finite range $\{-1, 1\}$. The set theorem requires an infinite set and cannot be applied to every sequence's range.
- (b) Every bounded infinite subset of $\mathbb{R}$ has an accumulation point in $\mathbb{R}$. A point L is an accumulation point of S if every deleted neighborhood of L meets S . If $(L - r, L + r)$ contained only finitely many points of $S \setminus \{L\}$, choose a positive radius smaller than r and all their distances to L (or any radius below r if there are none). Its deleted neighborhood would miss S , a contradiction.
- (c) The interval $(L - 1/k, L + 1/k)$ contains infinitely many distinct range values, whereas the first n_{k-1} terms supply only finitely many. Some value in that interval must therefore occur after index n_{k-1} ; choose such an index as n_k . The indices increase and the errors are less than $1/k$, proving subsequence convergence by Problem 1. The unselected terms need not approach L ; $(-1)^n$ is bounded but divergent.

Infinitely many terms need not mean infinitely many distinct values.

4. Sequential characterization of compactness: Theorem 5.1.

For $K \subseteq \mathbb{R}$, consider the property:

Every sequence in K has a subsequence converging to a point of K . (S)

- (a) State the open-cover definition of compactness and the Heine–Borel theorem for subsets of $\mathbb{R}$. If K is compact, use boundedness and Theorem 4.1 to obtain a convergent subsequence. Use closedness to show its limit belongs to K ; justify this step using the open complement of K . Explain why $(0, 1)$ fails (S) by considering $x_n = 1/(n + 1)$.
- (b) Conversely, assume (S). If K were unbounded, explain how to select $x_n \in K$ with $|x_n| > n$. Show that every subsequence of (x_n) is unbounded, and identify the fact about convergent sequences that contradicts (S). Conclude that K is bounded.
- (c) Still assuming (S), suppose K is not closed. Choose $x \in \text{cl}(K) \setminus K$ and use the neighborhood definition of closure to select $x_n \in K$ with $|x_n - x| < 1/n$. Find the limit of (x_n) and of every subsequence. Explain why (S) and uniqueness of limits give a contradiction, then finish the proof that K is compact.

Solution.

- (a) Compactness means every cover by open sets has a finite subcover. Heine–Borel says that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded. Boundedness and Theorem 4.1 give a subsequence with real limit L . If $L \notin K$, the open complement contains an interval about L disjoint from K , contradicting convergence of terms lying in K . In $(0, 1)$ the stated sequence and all its subsequences have limit zero, outside K , so (S) fails.
- (b) Unboundedness means no positive number bounds all absolute values, so each choice is possible. Since $n_k \geq k$, every subsequence satisfies $|x_{n_k}| > n_k \geq k$. Thus it is unbounded and cannot converge, since every convergent real sequence is bounded. This contradicts (S).
- (c) Closure means every neighborhood of x meets K , allowing the stated choices. The error bound gives $\lim_{n \rightarrow \infty} x_n = x$, and every subsequence also has limit x . Property (S) supplies one subsequence with a limit $y \in K$. Uniqueness gives $y = x$, contrary to $x \notin K$. Hence K is closed; together with boundedness, Heine–Borel proves compactness. For $K = \emptyset$, (S) holds vacuously and compactness holds directly.

Boundedness supplies a convergent selection; closedness retains its limit.

5. Why the ambient space matters: completeness of $\mathbb{R}$.

Let q_n be the decimal truncation of $\sqrt{2}$ after n decimal places. Thus $q_1 = 1.4$, $q_2 = 1.41$, $q_3 = 1.414$, and

$$q_n \in \mathbb{Q}, \quad 0 < \sqrt{2} - q_n < 10^{-n}.$$

- (a) Prove that $\lim_{n \rightarrow \infty} q_n = \sqrt{2}$ in $\mathbb{R}$. Use the triangle inequality to prove directly that (q_n) is Cauchy. Explain why the same Cauchy estimate holds when the terms are regarded as a sequence in $\mathbb{Q}$ with distance $|x - y|$.
- (b) Prove that no subsequence of (q_n) converges to a point of $\mathbb{Q}$. Define completeness and explain why this example shows that $\mathbb{Q}$ is not complete and that the bounded-sequence form of Bolzano–Weierstrass fails in $\mathbb{Q}$. In Problem 3, which existence step guarantees a real limit but cannot guarantee a rational one?
- (c) Let $K = [1, 2] \cap \mathbb{Q}$, viewed as a subset of $\mathbb{Q}$. It is bounded and closed in $\mathbb{Q}$: its complement in $\mathbb{Q}$ is open. Here, a set is open in $\mathbb{Q}$ if it is $\mathbb{Q} \cap O$ for some open $O \subseteq \mathbb{R}$. For $m \in \mathbb{N}$, define the sets open in $\mathbb{Q}$

$$U_m = \mathbb{Q} \cap \left((-\infty, \sqrt{2} - 1/m) \cup (\sqrt{2} + 1/m, \infty) \right).$$

Show that the U_m cover K . Next observe that $U_m \subseteq U_{m+1}$ and use (q_n) to show that no finite selection covers K . What does this show about extending Heine–Borel to $\mathbb{Q}$? Identify the earlier results in Problems 3 and 4 that depend on completeness of $\mathbb{R}$, rather than only on distance estimates.

Solution.

- (a) Given $\epsilon > 0$, choose N with $10^{-N} < \epsilon$. For $n \geq N$, $|q_n - \sqrt{2}| < 10^{-n} \leq 10^{-N} < \epsilon$. For $m, n \geq N$,

$$|q_n - q_m| \leq |q_n - \sqrt{2}| + |q_m - \sqrt{2}| < 2 \cdot 10^{-N}.$$

Choose N with $2 \cdot 10^{-N} < \epsilon$ to prove the Cauchy property. Distances between rational terms are unchanged when the ambient space changes.

- (b) Every subsequence has real limit $\sqrt{2}$. A rational limit would also be a real limit, contradicting uniqueness and the irrationality of $\sqrt{2}$. A space is complete if every Cauchy sequence in it converges to a point of that space. Thus $\mathbb{Q}$ is not complete. The bounded sequence (q_n) has no subsequence converging in $\mathbb{Q}$. In Problem 3(b), the set form of Bolzano–Weierstrass supplies an accumulation point in $\mathbb{R}$, not necessarily in $\mathbb{Q}$.
- (c) Each $q \in K$ differs from $\sqrt{2}$ by a positive distance, so choose m with $1/m < |q - \sqrt{2}|$ to obtain $q \in U_m$. The sets increase with m . A finite union is contained in U_M , where M is the largest selected index. Choose n with $10^{-n} < 1/M$. Then $q_n \in K$ but $|q_n - \sqrt{2}| < 1/M$, so $q_n \notin U_M$. Hence the cover has no finite subcover: closed and bounded in $\mathbb{Q}$ does not imply compact. Completeness, through the least-upper-bound property, underlies the real set form of Bolzano–Weierstrass and

Heine–Borel. These supply real accumulation points and compactness of closed bounded real sets. Selection and distance estimates alone do not supply missing rational limits.