Topology of the Real Numbers

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MATH 312

1 From Distance to Topology

The preceding notes constructed $\mathbb{R}$ as a complete ordered field and used absolute value to define the metric $d(x, y) = |x - y|$. A metric measures distance. Topology retains the information needed to discuss nearness: whether points of a set occur arbitrarily close to x, whether a small interval about x remains in a set, and whether a sequence approaches a point from within a set. These ideas will support limits and continuity.

Definition 1.1. For $x \in \mathbb{R}$ and $\varepsilon > 0$, the *open ε-neighborhood* of x is

$$N(x; \varepsilon) = \{y \in \mathbb{R} : |x - y| < \varepsilon\} = (x - \varepsilon, x + \varepsilon).$$

The *deleted ε-neighborhood* is

$$N^*(x; \varepsilon) = \{y \in \mathbb{R} : 0 < |x - y| < \varepsilon\} = N(x; \varepsilon) \setminus \{x\}.$$

Making ε smaller means looking more closely near x. A claim that holds in some neighborhood holds sufficiently close to x; a claim about every neighborhood holds no matter how closely we look.

Example 1.2. We have $N(3; 1/2) = (5/2, 7/2)$, whereas $N^*(3; 1/2) = (5/2, 3) \cup (3, 7/2)$. The deleted neighborhood is useful when we require a nearby point different from x.

Lemma 1.3 (Smaller neighborhoods). *If $y \in N(x; \varepsilon)$, then some $\delta > 0$ satisfies $N(y; \delta) \subseteq N(x; \varepsilon)$.*

The proof asks us to measure the room between y and the endpoints of $N(x; \varepsilon)$. We will develop this argument on the topology worksheet.

2 Points Relative to a Set

Fix $S \subseteq \mathbb{R}$. Notice which definitions require $x \in S$ and which permit $x \notin S$.

Definition 2.1.

(a) A point $x \in S$ is an *interior point* if some $\varepsilon > 0$ satisfies $N(x;\varepsilon) \subseteq S$.

(b) A point $x \in \mathbb{R}$ is a *boundary point* if every $\varepsilon > 0$ satisfies

$$N(x;\varepsilon) \cap S \neq \varnothing \quad \text{and} \quad N(x;\varepsilon) \cap S^c \neq \varnothing.$$

(c) A point $x \in \mathbb{R}$ is an *accumulation point* if every $\varepsilon > 0$ satisfies $N^*(x;\varepsilon) \cap S \neq \varnothing$.

(d) A point $x \in S$ is an *isolated point* if some $\varepsilon > 0$ satisfies $N^*(x;\varepsilon) \cap S = \varnothing$.

The deleted neighborhood in part (c) is essential: an ordinary neighborhood of a point $x \in S$ automatically meets S at x. We denote the sets of interior, boundary, and accumulation points by $\operatorname{int}(S)$, $\operatorname{bd}(S)$, and S', respectively.

Example 2.2 (A half-open interval). For $S = (0, 4]$,

$$\operatorname{int}(S) = (0, 4), \qquad \operatorname{bd}(S) = \{0, 4\}, \qquad S' = [0, 4].$$

The boundary point 0 is outside S, while the boundary point 4 is inside. Thus membership in S does not determine whether a point is a boundary point.

Example 2.3 (Isolated points). Every point of $\mathbb{Z}$ is isolated because $N^*(n; 1/2) \cap \mathbb{Z} = \varnothing$. Moreover,

$$\operatorname{int}(\mathbb{Z}) = \varnothing, \qquad \operatorname{bd}(\mathbb{Z}) = \mathbb{Z}, \qquad \mathbb{Z}' = \varnothing.$$

Example 2.4 (Dense sets). Every open interval contains both a rational and an irrational number. Consequently,

$$\operatorname{int}(\mathbb{Q}) = \varnothing, \qquad \operatorname{bd}(\mathbb{Q}) = \mathbb{R}, \qquad \mathbb{Q}' = \mathbb{R}.$$

Thus a set can have empty interior while being spread throughout $\mathbb{R}$.

Example 2.5 (A sequence as a set). Let $S = \{1/n : n \in \mathbb{N}\}$. Each $1/n$ is isolated. However, 0 is an accumulation point: given $\varepsilon > 0$, the Archimedean property provides n with $0 < 1/n < \varepsilon$. In fact, $S' = \{0\}$, even though $0 \notin S$.

Proposition 2.6. *If $x \in S$, then exactly one of the following holds: x is isolated in S, or x is an accumulation point of S.*

This proposition follows by writing the definition of an isolated point and carefully negating its quantifiers. We will supply the details on the topology worksheet.

3 Open and Closed Sets

Definition 3.1. A set $U \subseteq \mathbb{R}$ is *open* if every point of U is an interior point. A set $F \subseteq \mathbb{R}$ is *closed* if it contains all of its accumulation points; equivalently, $F' \subseteq F$.

Open and closed are not mutually exclusive labels. The sets $\varnothing$ and $\mathbb{R}$ are both open and closed. The interval $(0, 1)$ is open but not closed, $[0, 1]$ is closed but not open, and $(0, 1]$ is neither.

Example 3.2. Every open interval (a, b) is open. If $x \in (a, b)$, choose

$$\varepsilon = \tfrac{1}{2} \min\{x - a, b - x\} > 0.$$

Then $N(x; \varepsilon) \subseteq (a, b)$.

Theorem 3.3 (Complement characterization). *A set $F \subseteq \mathbb{R}$ is closed if and only if F^c is open.*

Proof. Suppose F is closed and $x \in F^c$. Then $x \notin F'$, so some deleted neighborhood of x misses F. Since $x \notin F$, the ordinary neighborhood also misses F and is contained in F^c. Thus F^c is open.

Conversely, suppose F^c is open and $x \in F'$. If $x \in F^c$, some neighborhood of x lies in F^c, contradicting the fact that every deleted neighborhood meets F. Hence $x \in F$, so F is closed. □

Corollary 3.4. *A set F is closed if and only if* $\mathrm{bd}(F) \subseteq F$. *A set U is open if and only if* $U \cap \mathrm{bd}(U) = \varnothing$.

Proof. If a boundary point of F lies outside F, it is an accumulation point of F. Conversely, an accumulation point outside F is a boundary point because each neighborhood meets F and also meets F^c at its center. Therefore, F is closed if and only if it $F' \subseteq F$, which is true if and only if $\mathrm{bd}(F) \subseteq F$.

Now, U is open if and only if U^c is closed, which is true if and only if $\mathrm{bd}((U^c)) \subseteq U^c$. By definition $\mathrm{bd}((U^c)) = \mathrm{bd}((U))$. Hence, U is open if and only if $\mathrm{bd}((U)) \subseteq U^c$, which is equivalent to $U \cap \mathrm{bd}((U)) = \varnothing$. □

Theorem 3.5 (Set operations). *Arbitrary unions and finite intersections of open sets are open. Arbitrary intersections and finite unions of closed sets are closed.*

Proof. If $x \in \bigcup_{\alpha \in A} U_\alpha$, then $x \in U_\beta$ for some β; a neighborhood contained in U_β lies in the union. If $x \in U_1 \cap \cdots \cap U_n$, choose $N(x; \varepsilon_i) \subseteq U_i$ and use $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. The closed-set claims follow by taking complements, using De Morgan's laws and Theorem 3.3. □

The finiteness conditions matter:

$$\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\} \quad \text{and} \quad \bigcup_{n=1}^{\infty} [1/n, 1] = (0, 1].$$

4 Closure and Boundary

Definition 4.1. The *closure* of $S \subseteq \mathbb{R}$ is

$$\mathrm{cl}(S) = S \cup S'.$$

Thus closure adjoins every missing accumulation point.

Theorem 4.2 (Neighborhood characterization). *For $x \in \mathbb{R}$, we have $x \in \mathrm{cl}(S)$ if and only if every neighborhood of x intersects S.*

Proof. If $x \in S$, every neighborhood meets S at x; if $x \in S'$, every deleted neighborhood meets S. Conversely, suppose every neighborhood meets S. If $x \in S$, then $x \in \mathrm{cl}(S)$. If $x \notin S$, every point in the intersection differs from x, so every deleted neighborhood meets S and $x \in S'$. □

Theorem 4.3 (Properties of closure). *For every $S \subseteq \mathbb{R}$:*

(a) $\mathrm{cl}(S)$ *is closed;*

(b) $\mathrm{cl}(S) = S \cup \mathrm{bd}(S)$*;*

(c) S *is closed if and only if* $S = \mathrm{cl}(S)$*;*

(d) $\mathrm{cl}(S)$ *is the smallest closed set containing* S*.*

Proof. If $x \notin \mathrm{cl}(S)$, Theorem 4.2 gives a neighborhood $N(x;\varepsilon)$ disjoint from S. Lemma 1.3 shows that each point of this neighborhood has its own neighborhood disjoint from S. Hence $(\mathrm{cl}(S))^c$ is open, proving (a).

For (b), a point of $\mathrm{cl}(S) \setminus S$ has every neighborhood meet S and meet S^c at its center, so it is a boundary point. Conversely, every boundary point has every neighborhood meet S, and therefore belongs to $\mathrm{cl}(S)$. Part (c) follows from $\mathrm{cl}(S) = S \cup S'$ and part (a).

For (d), suppose F is closed and $S \subseteq F$. Every neighborhood of a point in $\mathrm{cl}(S)$ meets S, hence meets F. Thus $\mathrm{cl}(S) \subseteq \mathrm{cl}(F) = F$. □

Corollary 4.4. *For every $S \subseteq \mathbb{R}$,*

$$\mathrm{bd}(S) = \mathrm{cl}(S) \cap \mathrm{cl}((S^c)) = \mathrm{cl}(S) \setminus \mathrm{int}(S).$$

Consequently, $\mathrm{bd}(S) = \mathrm{bd}((S^c))$*, and every boundary is closed.*

5 Sequential Characterizations

Theorem 5.1 (Accumulation points and sequences). *A point x is an accumulation point of S if and only if a sequence (s_n) in $S \setminus \{x\}$ converges to x.*

Proof. If $x \in S'$, choose $s_n \in N^*(x; 1/n) \cap S$. Then $0 < |s_n - x| < 1/n$, so $s_n \to x$. Conversely, if $s_n \to x$ and $s_n \neq x$, then for every $\varepsilon > 0$, some s_n belongs to $N^*(x; \varepsilon) \cap S$. □

Theorem 5.2 (Closed sets and convergent sequences). *A set $F \subseteq \mathbb{R}$ is closed if and only if every convergent sequence of points in F has its limit in F.*

Proof. Suppose F is closed, $x_n \in F$, and $x_n \to x$. If some term equals x, then $x \in F$. Otherwise, after passing to a subsequence if necessary, Theorem 5.1 shows $x \in F' \subseteq F$. Conversely, if every convergent sequence in F has its limit in F, then Theorem 5.1 shows that every point of F' belongs to F. Hence F is closed. □

Corollary 5.3. *A point x belongs to $\mathrm{cl}(S)$ if and only if some sequence in S converges to x.*

Proof. If $x \in S$, use the constant sequence. If $x \in \mathrm{cl}(S) \setminus S$, use Theorem 5.1. Conversely, if a sequence in S converges to x, then every neighborhood of x meets S, so Theorem 4.2 gives $x \in \mathrm{cl}(S)$. □

6 A Working Summary

- $x \in \mathrm{int}(S)$: some neighborhood stays inside S.
- $x \in \mathrm{bd}(S)$: every neighborhood sees both S and S^c.
- $x \in S'$: other points of S occur arbitrarily close to x, equivalently, a sequence in $S \setminus \{x\}$ converges to x.
- $x \in \mathrm{cl}(S)$: every neighborhood meets S, equivalently, a sequence in S converges to x.
- An open set contains a neighborhood of each of its points. A closed set contains every limit point forced upon it by nearby points of the set.

These notions turn the metric and order structure of $\mathbb{R}$ into the language needed for limits and continuity.