

Topology and Compactness Quiz

Math 312: Concepts of Real Analysis

Answer Key

1. Topological definitions and closure.

- (a) State the definitions of an interior point and an accumulation point of a set $S \subseteq \mathbb{R}$, including all quantifiers.
- (b) State the definitions of an open set and a closed set in $\mathbb{R}$.
- (c) Let

$$S = \left\{ \frac{2}{n} : n \in \mathbb{N} \right\} \cup \{3\}, \quad \mathbb{N} = \{1, 2, 3, \dots\}.$$

Is S closed? Is $\text{cl}(S)$ compact? Justify both answers.

Solution. A point $x \in S$ is an interior point of S if there exists $\varepsilon > 0$ such that $N(x; \varepsilon) \subseteq S$. A point $x \in \mathbb{R}$ is an accumulation point of S if for every $\varepsilon > 0$ there exists $s \in S$ such that $0 < |s - x| < \varepsilon$.

A set is open if every point of the set is an interior point. A set is closed if it contains all of its accumulation points.

The set S is not closed: $0 \notin S$, but for every $\varepsilon > 0$, choosing $n > 2/\varepsilon$ gives $0 < 2/n < \varepsilon$, so $0 \in S'$. Its closure is compact. Indeed, closure is closed, and $S \subseteq [0, 3]$ implies $\text{cl}(S) \subseteq [0, 3]$ because $[0, 3]$ is closed. Thus $\text{cl}(S)$ is closed and bounded, hence compact by Heine–Borel.

2. Compactness and open covers.

- (a) State the definition of a compact set using open covers and finite subcovers. Make clear which sets may be selected for a subcover.
- (b) For $n \in \mathbb{N}$, let $U_n = (1/(n+1), 2)$. Show that $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an open cover of $(0, 1]$.
- (c) Does this cover have a finite subcover? Is $(0, 1]$ compact? Justify your answers.

Solution. A set $K \subseteq \mathbb{R}$ is compact if every collection $\mathcal{U}$ of open subsets of $\mathbb{R}$ with $K \subseteq \bigcup_{U \in \mathcal{U}} U$ has a finite subcollection $\mathcal{V} \subseteq \mathcal{U}$ satisfying $K \subseteq \bigcup_{U \in \mathcal{V}} U$. A subcover must use sets from the original cover.

Each U_n is open. If $0 < x \leq 1$, choose $n \in \mathbb{N}$ sufficiently large that $1/(n+1) < x$. Then $x \in U_n$, so $\mathcal{U}$ covers $(0, 1]$.

Consider any nonempty finite subcollection $U_{n_1}, \dots, U_{n_k}$ and let $N = \max\{n_1, \dots, n_k\}$. The intervals are nested, so their union is $U_N = (1/(N+1), 2)$. The point $1/(N+1)$ belongs to $(0, 1]$ but not to this union. The empty subcollection also fails to cover $(0, 1]$. Thus this open cover has no finite subcover, proving noncompactness.

3. Heine–Borel and extreme points.

- (a) State the Heine–Borel theorem for subsets of $\mathbb{R}$.
- (b) Let $K \subseteq \mathbb{R}$ be compact and nonempty. Prove that $\sup K$ exists and belongs to K .

Solution. Heine–Borel states that $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

By Heine–Borel, K is bounded and closed. Nonemptiness and completeness give $s = \sup K \in \mathbb{R}$. For every $\varepsilon > 0$, the number $s - \varepsilon$ is not an upper bound, so some $y \in K$ satisfies $s - \varepsilon < y \leq s$. Thus every neighborhood of s meets K , so $s \in \text{cl}(K) = K$. In particular, $s = \max K$.