

Topology Worksheet

Real Analysis

Answer Key

1. Working from the definitions.

Consider

$$A = (-1, 2] \cup \{3\} \quad \text{and} \quad B = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

- (a) Determine $\text{int}(A)$, $\text{bd}(A)$, A' , and $\text{cl}(A)$. Identify every isolated point of A .
- (b) Determine $\text{int}(B)$, $\text{bd}(B)$, B' , and $\text{cl}(B)$. Identify every isolated point of B .
- (c) For each set, decide whether it is open, closed, both, or neither. Justify your conclusions from the definitions.

Solution.

- (a) $\text{int}(A) = (-1, 2)$, $\text{bd}(A) = \{-1, 2, 3\}$, $A' = [-1, 2]$, and $\text{cl}(A) = [-1, 2] \cup \{3\}$. Each point of $(-1, 2)$ has a neighborhood contained in A . Every neighborhood of either endpoint meets both A and A^c , and contains a point of $(-1, 2)$ different from its center. The point 3 is isolated: $N(3; 1/2) \cap A = \{3\}$. It is also a boundary point, since every neighborhood of 3 contains points outside A . All remaining points lie outside the closure.
- (b) $\text{int}(B) = \emptyset$, $\text{bd}(B) = B \cup \{0\}$, $B' = \{0\}$, and $\text{cl}(B) = B \cup \{0\}$. Every $1/n$ is isolated: for $n \geq 2$, take half the smaller of $1/(n-1) - 1/n$ and $1/n - 1/(n+1)$; at 1, take radius $1/4$. Every neighborhood of 0 contains $1/n$ for sufficiently large n , so 0 is an accumulation point. For any $x > 0$, all but finitely many points of B lie below $x/2$. A small deleted neighborhood of x avoids the finitely many remaining points. A neighborhood of any $x < 0$ avoids B . Thus there are no other accumulation points. No neighborhood is contained in B , since every interval contains an irrational number. Hence every point of $B \cup \{0\}$ is a boundary point.
- (c) Both sets are neither open nor closed. The point $2 \in A$ is not interior, and the accumulation point -1 is missing from A . No point of B is interior, and its accumulation point 0 is missing.

Begin by asking what every sufficiently small neighborhood of a typical point must contain.

2. Closed sets and open complements.

Prove that $F \subseteq \mathbb{R}$ is closed if and only if F^c is open. Your proof must establish both directions.

- (a) Assume F is closed and choose $x \in F^c$. Why is $x \notin F'$? Translate this into the existence of a deleted neighborhood that misses F . Why may “deleted” then be removed?
- (b) Assume F^c is open and choose $x \in F'$. Suppose, toward a contradiction, that $x \in F^c$. Compare the neighborhood from openness with the neighborhoods required by $x \in F'$.
- (c) Apply the theorem to explain why $[a, b]$ is closed without determining its accumulation points directly.

Solution.

- (a) Closedness means $F' \subseteq F$. If $x \in F^c$, then $x \notin F'$, so some $\varepsilon > 0$ satisfies $(N(x; \varepsilon) \setminus \{x\}) \cap F = \emptyset$. Since $x \notin F$, the full neighborhood also misses F . Thus $N(x; \varepsilon) \subseteq F^c$, proving that F^c is open.
- (b) If $x \in F' \cap F^c$, openness gives $\varepsilon > 0$ with $N(x; \varepsilon) \subseteq F^c$. This contradicts the requirement that every deleted neighborhood of x meet F . Therefore $F' \subseteq F$, so F is closed.
- (c) For $a \leq b$, $[a, b]^c = (-\infty, a) \cup (b, \infty)$. If $x < a$, radius $(a-x)/2$ gives a neighborhood in the complement; if $x > b$, use $(x-b)/2$. Thus the complement is open and $[a, b]$ is closed. If $a > b$, the interval is empty and is also closed.

Your proof should make clear where each hypothesis is used.

3. Combining open and closed sets.

- (a) Let $U_1, \dots, U_n$ be open subsets of $\mathbb{R}$. Prove that $U_1 \cap \dots \cap U_n$ is open. Explain why

$$\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$$

is the key choice.

- (b) Explain why the argument in part (a) does not extend directly to an infinite collection. Prove that

$$\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\},$$

and conclude that an infinite intersection of open sets need not be open.

- (c) Use part (a), the open-complement characterization, and De Morgan's law to prove that a finite union of closed sets is closed.

Solution.

- (a) For $x \in \bigcap_{i=1}^n U_i$, choose $\varepsilon_i > 0$ with $N(x; \varepsilon_i) \subseteq U_i$. The minimum $\varepsilon = \min_i \varepsilon_i$ is positive because there are finitely many radii, and $N(x; \varepsilon) \subseteq \bigcap_i U_i$. Thus the intersection is open. If it is empty, openness holds vacuously.
- (b) Infinitely many positive radii need not have a positive lower bound. Here 0 belongs to every $(-1/n, 1/n)$. Conversely, if $x \neq 0$, the Archimedean property gives $n > 1/|x|$, so $1/n < |x|$ and $x \notin (-1/n, 1/n)$. Therefore the intersection is exactly $\{0\}$. Every neighborhood of 0 contains nonzero points, so $\{0\}$ is not open.
- (c) If $F_1, \dots, F_n$ are closed, each F_i^c is open and

$$(F_1 \cup \dots \cup F_n)^c = F_1^c \cap \dots \cap F_n^c$$

is open by part (a). The open-complement characterization implies that $F_1 \cup \dots \cup F_n$ is closed.

For part (b), establish both set inclusions before drawing the topological conclusion.

4. Comparing the point types.

For each statement, decide whether it is true or false. If it is true, prove it directly from the definitions. If it is false, give a counterexample and explain why it works.

- (a) Every interior point of a set $S \subseteq \mathbb{R}$ is an accumulation point of S .
- (b) Every accumulation point of S is an interior point of S .
- (c) Every isolated point of S is a boundary point of S .
- (d) Every boundary point of S is an accumulation point of S .

Then explain in one or two sentences why parts (a) and (b) establish the claim: every interior point is an accumulation point, but not conversely.

Solution.

- (a) True. If $N(x; r) \subseteq S$ for some $r > 0$, then for any $\varepsilon > 0$, set $t = \min\{r, \varepsilon\}/2$. The point $x + t$ belongs to $S \cap N(x; \varepsilon)$ and differs from x . Hence $x \in S'$.
- (b) False. For $S = (0, 1)$, every deleted neighborhood of 0 meets S , but 0 is not an interior point (indeed, $0 \notin S$).
- (c) True. An isolated point x belongs to S , and some $N(x; r)$ meets S only at x . Every neighborhood of x meets S at x . For any $\varepsilon > 0$, the point $x + \min\{r, \varepsilon\}/2$ is in that neighborhood and outside S . Thus every neighborhood meets both S and S^c .
- (d) False. For $S = \{0\}$, every neighborhood of 0 meets both S and S^c , so 0 is a boundary point. Every deleted neighborhood misses S , so 0 is not an accumulation point.

Part (a) proves that interior points are always accumulation points. Part (b) gives an accumulation point that is not interior, so the converse fails.

Useful examples include intervals, finite sets, and $\{1/n : n \in \mathbb{N}\}$. For part (a), an open interval contains a point different from its center.

5. Neighborhoods and logical negation.

Let $S \subseteq \mathbb{R}$.

- (a) Suppose $x \in S$. Write the definition that x is isolated in S , including all quantifiers. Negate the statement carefully and show that the result says exactly that x is an accumulation point of S . Explain why exactly one possibility holds.
- (b) Suppose $y \in N(x; \varepsilon)$. Find an explicit $\delta > 0$ such that

$$N(y; \delta) \subseteq N(x; \varepsilon),$$

and prove the inclusion using the triangle inequality.

- (c) Use part (b) to prove that every neighborhood $N(x; \varepsilon)$ is open.

Solution.

- (a) Under the assumption $x \in S$, isolation means

$$\exists \varepsilon > 0 \forall z \in \mathbb{R}, \quad (0 < |z - x| < \varepsilon \implies z \notin S).$$

Its negation is

$$\forall \varepsilon > 0 \exists z \in \mathbb{R}, \quad 0 < |z - x| < \varepsilon \text{ and } z \in S,$$

which is exactly the definition of $x \in S'$. A statement and its negation cannot both hold, and one must hold; therefore exactly one of isolation and accumulation holds for $x \in S$.

- (b) Set $\delta = \varepsilon - |y - x| > 0$. If $z \in N(y; \delta)$, then the triangle inequality gives

$$|z - x| \leq |z - y| + |y - x| < \delta + |y - x| = \varepsilon.$$

Thus $z \in N(x; \varepsilon)$, proving the inclusion.

- (c) For every $y \in N(x; \varepsilon)$, part (b) supplies $\delta > 0$ such that $N(y; \delta) \subseteq N(x; \varepsilon)$. Every point of the neighborhood is therefore interior, so the neighborhood is open.

Hint for (b): Measure the distance from y to the boundary of $N(x; \varepsilon)$.