# Uniform Continuity

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## 1 Uniform Continuity

Let $S \subseteq \mathbb{R}$ and $f\colon S \to \mathbb{R}$. We say that $f$ is *continuous on* $S$ if $f$ is continuous at every $c \in S$, that is, for each $c \in S$ and for every $\epsilon \in \mathbb{R}_{>0}$ there is a $\delta \in \mathbb{R}_{>0}$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$ and $x \in S$. It is important to note that the choice of $\delta$ may depend on both $\epsilon$ and $c$. If it happens that for each $\epsilon \in \mathbb{R}_{>0}$ there is a $\delta \in \mathbb{R}_{>0}$ that works for all $c \in S$, then $f$ is said to be uniformly continuous on $S$. Specifically, we say that $f$ is *uniformly continuous on* $S$ if for all $\epsilon \in \mathbb{R}_{>0}$ there is a $\delta \in \mathbb{R}_{>0}$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ and $x, y \in S$.

As an example, we will show that $f(x) = 2x$ is uniformly continuous on $\mathbb{R}$. To this end, let $\epsilon \in \mathbb{R}_{>0}$ and define $\delta = \frac{\epsilon}{2}$ Then, for any $x, y \in \mathbb{R}$, we have

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| = 2\,|x - y| < 2\delta = \epsilon.$$

In contrast, we will show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. To this end, let $\epsilon = 1$. Then, for any $\delta \in \mathbb{R}_{>0}$, define $x = \frac{1}{\delta}$ and $y = \frac{1}{\delta} + \frac{\delta}{2}$. So, $|x - y| < \delta$ and

$$\begin{aligned}
|f(x) - f(y)| &= \left|x^2 - y^2\right| \\
&= |x + y|\,|x - y| \\
&= \left(\frac{2}{\delta} + \frac{\delta}{2}\right)\left(\frac{\delta}{2}\right) \\
&\geq \frac{2}{\delta}\frac{\delta}{2} = 1.
\end{aligned}$$

In the previous example, $|x - y|$ is made small while $|x + y|$ is made large. This can't happend on bounded domains. For example, $f(x) = x^2$ is uniformly continuous on $[0, 1]$. Let $\epsilon \in \mathbb{R}_{>0}$ and define $\delta = \frac{\epsilon}{2}$. Then, for any $x, y \in [0, 1]$, we have

$$|x - y| < \delta \Rightarrow \left|x^2 - y^2\right| = |x + y|\,|x - y| \leq 2\,|x - y| < 2\delta = \epsilon.$$

In the previous example, we had a continuous function $f(x) = x^2$ over a compact domain $[0, 1]$. The following theorem shows that this is always sufficient for uniform continuity.

**Theorem 1.1.** *Suppose that $S \subseteq \mathbb{R}$ is compact and $f\colon S \to \mathbb{R}$ is continuous. Then, $f$ is uniformly continuous on $S$.*

*Proof.* Let $\epsilon \in \mathbb{R}_{>0}$. For each $c \in S$, there is a $\delta_c \in \mathbb{R}_{>0}$ such that $|f(x) - f(c)| < \frac{\epsilon}{2}$ whenever $|x - c| < \delta_c$ and $x \in S$. Note that $\mathcal{F} = \left\{N(c; \frac{\delta_c}{2})\colon c \in S\right\}$ is an open cover for $S$. Since $S$ is compact, there is a finite subcover of $\mathcal{F}$; that is, there is a $k \in \mathbb{N}$ such that

$$S \subseteq \bigcup_{i=1}^{k} N(c_i; \frac{\delta_{c_i}}{2}).$$

Now, define $\delta = \min\left\{\frac{\delta_{c_i}}{2}\colon 1 \leq i \leq k\right\}$. Then, let $x, y \in S$ such that $|x - y| < \delta$. Since $x \in S$, there is an $i \in \{1, \ldots, k\}$ such that $x \in N(c_i; \frac{\delta_{c_i}}{2})$. Thus, $|f(x) - f(c_i)| < \frac{\epsilon}{2}$. Furthermore, since $|x - y| < \delta \leq \frac{\delta_{c_i}}{2}$, we have

$$|y - c_i| \leq |y - x| + |x - c_i| < \delta_{c_i}.$$

So, $|f(y) - f(c_i)| < \frac{\epsilon}{2}$. Therefore,

$$|f(x) - f(y)| \leq |f(x) - f(c_i)| + |f(c_i) - f(y)| < \epsilon.$$

$\square$

Theorem 1.1 implies that over compact domains, the properties of continuous functions apply to uniformly continuous functions. For instance, we have the following corollary.

**Corollary 1.2.** *Let $S \subseteq \mathbb{R}$ be compact. Also, let $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ be continuous. Then, $f$ and $g$ are uniformly continuous on $S$. Moreover,*

*(a) $f + g$ is uniformly continuous on $S$.*

*(b) $kf$ is uniformly continuous on $S$, for all $k \in \mathbb{R}$.*

*(c) $f \cdot g$ is uniformly continuous on $S$.*

*(d) $f/g$ is uniformly continuous on $S$, provided $g$ is non-zero on $S$.*

The following example demonstrates what can happen for continuous functions over non-compact domains. In particular, we claim that $f(x) = 1/x$ is not uniformly continuous on $(0,1)$. To this end, let $\epsilon = 1$. Then, for any $\delta \in \mathbb{R}_{>0}$ let $x, y \in (0,1)$ such that $x < \delta$ and $y = x/2$. Note that since $x, y \in (0,1)$ we can assume that $\delta < 1$. Furthermore, $|x - y| = \frac{x}{2} < \frac{\delta}{2} < \delta$. However,

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{1}{x} > \frac{1}{\delta} > 1.$$