Uniform Continuity

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MATH 312

1 From Continuity to Uniform Continuity

Continuity at c controls output error by restricting how far an input may move from c. The permitted input error can change as c moves through the domain. Uniform continuity asks whether one input tolerance controls every pair of nearby domain points.

Definition 1.1 (Uniform continuity). Let $S \subseteq \mathbb{R}$ and $f\colon S \to \mathbb{R}$. The function f is *uniformly continuous on* S if, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in S \text{ and } |x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \epsilon.$$

The same δ must work for *all* $x, y \in S$. It may depend on ϵ, the function, and the domain, but not on the selected inputs. Compare the order of the quantifiers:

$$\begin{aligned} \text{Continuity on } S: &\quad \forall c \in S\ \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in S, \\ \text{Uniform continuity on } S: &\quad \forall \epsilon > 0\ \exists \delta > 0\ \forall c \in S\ \forall x \in S. \end{aligned}$$

In both lines the final condition is $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. Moving the choice of δ before the choice of c makes a stronger demand.

Proposition 1.2. *Every uniformly continuous function on S is continuous on S. A restriction of a uniformly continuous function to any subset of S is uniformly continuous there.*

Proof. For continuity, fix $c \in S$ and use the uniform choice with $y = c$. For a restriction, the same δ still controls every pair in the smaller domain. □

Example 1.1 (A global estimate). Let $f(x) = 3x - 2$ on $\mathbb{R}$. Given $\epsilon > 0$, choose $\delta = \epsilon/3$. For every $x, y \in \mathbb{R}$,

$$|x - y| < \delta \implies |f(x) - f(y)| = 3|x - y| < 3\delta = \epsilon.$$

The estimate involves only the distance between the inputs, so the location of those inputs does not affect the choice.

Uniform continuity is a property of a function *on a specified domain*. Changing the domain can change the answer, even when the formula stays the same.

2 Estimates and the Role of the Domain

Example 2.1 (The square function on a bounded interval). For $f(x) = x^2$ and $x, y \in [-M, M]$, where $M > 0$,

$$\left|x^2 - y^2\right| = |x + y||x - y| \leq 2M|x - y|.$$

Thus $\delta = \epsilon/(2M)$ proves uniform continuity on $[-M, M]$, and on every subset of this interval, including $(-M, M)$.

On all of $\mathbb{R}$, the factor $|x + y|$ has no fixed bound. This does not by itself prove failure of uniform continuity, but it suggests looking for nearby inputs far from the origin.

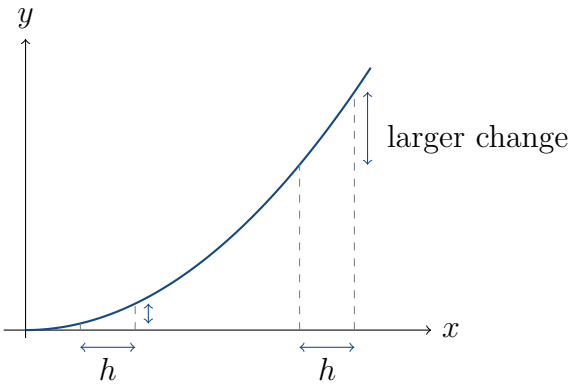


Figure 1. Equal input changes can produce larger output changes as the inputs move.

Definition 2.1 (Lipschitz condition). A function $f\colon S \to \mathbb{R}$ is *Lipschitz* if there is $L \geq 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in S$.

If $L > 0$, choose $\delta = \epsilon/L$ to obtain uniform continuity. If $L = 0$, the function is constant on its domain and any $\delta > 0$ works.

Example 2.2 (Uniform continuity without a Lipschitz bound). For $x \geq y \geq 0$, we have $(\sqrt{x} - \sqrt{y})^2 \leq (\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) = x - y$. Interchanging x, y when necessary gives

$$\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|}.$$

Therefore $\delta = \epsilon^2$ proves uniform continuity of $\sqrt{x}$ on $[0, \infty)$. It is not Lipschitz there: comparing $x > 0$ to 0 would require $1/\sqrt{x} \leq L$ for all $x > 0$.

Example 2.3 (A direct estimate for absolute value). The reverse triangle inequality gives

$$\left||x| - |y|\right| \leq |x - y| \qquad (x, y \in \mathbb{R}).$$

Thus $f(x) = |x|$ is Lipschitz with $L = 1$ and uniformly continuous on $\mathbb{R}$, using $\delta = \epsilon$. The corner at zero causes no difficulty: uniform continuity is about differences of function values and does not require differentiability.

3 How Uniform Continuity Can Fail

Negating the definition requires a *single* positive output error that defeats every proposed input tolerance.

Proposition 3.1 (Negation and sequential criterion). *A function $f\colon S \to \mathbb{R}$ is not uniformly continuous on S if and only if there exists $\epsilon_0 > 0$ such that, for every $\delta > 0$, there exist $x, y \in S$ satisfying*

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| \geq \epsilon_0.$$

Equivalently, there exist sequences $(x_n), (y_n)$ in S and $\epsilon_0 > 0$ such that

$$\lim_{n\to\infty} |x_n - y_n| = 0, \qquad |f(x_n) - f(y_n)| \geq \epsilon_0 \quad (n \in \mathbb{N}).$$

Proof. The first statement is the logical negation of the definition. If it holds, choose a violating pair for $\delta = 1/n$ for each n. Their distances tend to zero. Conversely, given these sequences and any $\delta > 0$, choose n with $|x_n - y_n| < \delta$. This pair violates the uniform continuity condition for ϵ_0. □

Thus uniform continuity is also equivalent to the statement that *every* pair of sequences with input distances tending to zero has output distances tending to zero. Neither input sequence is required to converge. Indeed, uniform continuity applied to an arbitrary $\epsilon > 0$ controls all pairs with sufficiently small input distance. Conversely, failure supplies the violating sequences in Proposition 3.1.

Example 3.1 (Failure at infinity). The continuous function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. Take $x_n = n$ and $y_n = n + 1/n$. Then

$$|x_n - y_n| = 1/n, \qquad |f(y_n) - f(x_n)| = 2 + 1/n^2 \geq 2.$$

Proposition 3.1 applies with $\epsilon_0 = 2$. Both input sequences escape to infinity while becoming arbitrarily close to one another.

Example 3.2 (Failure near a missing endpoint). The function $g(x) = 1/x$ is continuous on $(0, 1)$. Let $x_n = 1/(n + 1)$ and $y_n = 1/(n + 2)$. Both lie in the domain, and

$$\lim_{n\to\infty} |x_n - y_n| = \lim_{n\to\infty} \frac{1}{(n + 1)(n + 2)} = 0, \qquad |g(x_n) - g(y_n)| = 1.$$

Hence g is not uniformly continuous on $(0, 1)$, although its domain is bounded. Both sequences approach 0, where continuity on $(0, 1)$ gives no information.

By contrast, on $[a, \infty)$ with $a > 0$, the estimate $|1/x - 1/y| \leq |x - y|/a^2$ proves uniform continuity. Keeping the inputs away from zero changes the conclusion.

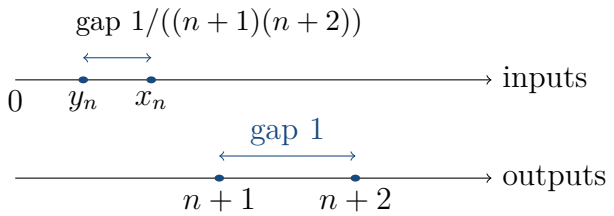


Figure 2. For $1/x$ on $(0, 1)$, nearby inputs retain separated outputs (schematic).

4 Compactness Gives Uniform Continuity

Theorem 4.1 (Heine–Cantor theorem). *If $K \subseteq \mathbb{R}$ is compact and $f\colon K \to \mathbb{R}$ is continuous, then f is uniformly continuous on K.*

Proof. The claim is immediate if K is empty. Otherwise, suppose uniform continuity fails. Proposition 3.1 supplies $\epsilon_0 > 0$ and sequences $(x_n), (y_n)$ in K with

$$|x_n - y_n| < 1/n, \qquad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

By compactness, some subsequence (x_{n_k}) converges to $c \in K$. The triangle inequality gives

$$|y_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| < 1/n_k + |x_{n_k} - c|.$$

Both terms on the right tend to zero, so $\lim_{k\to\infty} y_{n_k} = c$ as well. Continuity at c now gives

$$\lim_{k\to\infty} f(x_{n_k}) = f(c) = \lim_{k\to\infty} f(y_{n_k}).$$

Consequently their output differences tend to zero, contradicting the lower bound ϵ_0 for every k. □

Boundedness supplies a convergent subsequence through Bolzano–Weierstrass; closedness keeps its limit in K; continuity then controls both outputs at that common limit. Notice that we select a subsequence of (x_n) and then use the *same indices* for (y_n). Two independently selected subsequences need not approach the same point. This is another way completeness of $\mathbb{R}$ turns local information into a conclusion on the whole domain.

4.1 The Finite-Subcover Viewpoint

There is also a direct way to see how local tolerances produce one global tolerance. Assume K is nonempty and fix $\epsilon > 0$. At each $c \in K$, choose $r_c > 0$ so that

$$x \in K, \quad |x - c| < r_c \quad \Longrightarrow \quad |f(x) - f(c)| < \epsilon/2.$$

The neighborhoods $N(c; r_c/2)$ cover K. Compactness supplies a finite subcover with centers $c_1, \ldots, c_m$. Set

$$\delta = \min_{1\leq i\leq m} r_{c_i}/2 > 0.$$

If $x, y \in K$ and $|x - y| < \delta$, choose i with $|x - c_i| < r_{c_i}/2$. The triangle inequality gives

$$|y - c_i| \leq |y - x| + |x - c_i| < \delta + r_{c_i}/2 \leq r_{c_i}.$$

The smaller covering neighborhoods ensure that y remains in the larger neighborhood where continuity at c_i applies. Thus

$$|f(x) - f(y)| \leq |f(x) - f(c_i)| + |f(c_i) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Taking the minimum of *finitely many* positive radii ensures positivity. The infimum of all local radii could instead be zero; compactness is what allows the finite selection.

5 Using the Compactness Theorem

Example 5.1 (A compact domain need not be an interval). Let $K = [-2, -1] \cup [1, 2]$. The function $f(x) = 1/x$ is continuous on K, and K is closed and bounded. Therefore f is uniformly continuous on K. A gap in the domain does not obstruct the theorem.

Example 5.2 (A continuous extension supplies uniform control). The function $f(x) = x^2$ on $(0, 1)$ is the restriction of the continuous function $F(x) = x^2$ on $[0, 1]$. Heine–Cantor gives uniform continuity of F, and restriction gives uniform continuity of f.

The domain $(0, 1)$ is not compact. Compactness is sufficient for every continuous function on a domain to be uniformly continuous; an individual function can be uniformly continuous without a compact domain.

Example 5.3 (An unbounded uniformly continuous function). The identity $f(x) = x$ is uniformly continuous on $\mathbb{R}$, using $\delta = \epsilon$, but is unbounded. Uniform continuity controls changes in output for nearby inputs; it does not bound all output values.

Corollary 5.1 (Algebra on a compact domain). *Let $K \subseteq \mathbb{R}$ be compact, and let $f, g \colon K \to \mathbb{R}$ be continuous. Then $f + g$, af for $a \in \mathbb{R}$, and fg are uniformly continuous on K. If $g(x) \neq 0$ for every $x \in K$, then f/g is uniformly continuous on K.*

Proof. Each resulting function is continuous on K by the continuity rules. Apply Theorem 4.1 to each one. □

Remark 5.2 (Products on general domains). The product of two uniformly continuous functions need not be uniformly continuous: $f(x) = g(x) = x$ on $\mathbb{R}$ are uniformly continuous, but their product x^2 is not.

If both functions are bounded as well as uniformly continuous, their product is uniformly continuous. Indeed, choose $M \geq 1$ bounding both $|f|$ and $|g|$. Then

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\leq |f(x)||g(x) - g(y)| \\ &\quad + |g(y)||f(x) - f(y)| \\ &\leq M\Big(|g(x) - g(y)| + |f(x) - f(y)|\Big). \end{aligned}$$

Choose the two uniform tolerances for output error $\epsilon/(2M)$, and take their minimum. Compactness provides the boundedness automatically for continuous functions.

Example 5.4 (Bounded outputs do not guarantee uniform continuity). Consider $h(x) = \sin(1/x)$ on $(0, 1)$. This function is continuous and satisfies $|h(x)| \leq 1$. Nevertheless, set

$$x_n = \frac{1}{2\pi n + \pi/2}, \qquad y_n = \frac{1}{2\pi n + 3\pi/2}.$$

Both input sequences converge to zero, so their distances tend to zero. But $h(x_n) = 1$ and $h(y_n) = -1$, giving output distance 2 for every n. Hence h is not uniformly continuous. Rapid oscillation near a missing endpoint can defeat uniform control even when the outputs stay bounded.

6 Uniform Continuity and Cauchy Sequences

Continuity preserves the limit of a sequence converging to a point *in the domain*. A Cauchy sequence need not have its limit in the domain. Uniform continuity still controls all sufficiently late pairs of terms.

Theorem 6.1 (Preservation of Cauchy sequences). *Let $f\colon S \to \mathbb{R}$ be uniformly continuous. If (x_n) is a Cauchy sequence in S, then $(f(x_n))$ is Cauchy in $\mathbb{R}$.*

Proof. Let $\epsilon > 0$. Uniform continuity gives $\delta > 0$ such that $x, y \in S$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$. Since (x_n) is Cauchy, there is $N \in \mathbb{N}$ such that

$$m, n \geq N \quad \Longrightarrow \quad |x_m - x_n| < \delta.$$

For these same indices, $|f(x_m) - f(x_n)| < \epsilon$, which is the Cauchy condition for the output sequence. □

Example 6.1 (Continuity alone is insufficient). On $S = (0, 1)$, take $f(x) = 1/x$ and $x_n = 1/(n + 1)$. The input sequence converges to zero and is therefore Cauchy. Its image $f(x_n) = n + 1$ is unbounded, hence not Cauchy. Continuity on S does not apply at the missing endpoint 0.

Corollary 6.2 (A limit at a missing endpoint). *If $a < b$ are finite and $f\colon (a, b) \to \mathbb{R}$ is uniformly continuous, then $\lim_{x\to a^+} f(x)$ exists as a finite real number. The analogous conclusion holds at b.*

Proof. Set $t_n = a + (b - a)/(n + 1)$. This is a Cauchy sequence in (a, b), so $(f(t_n))$ is Cauchy. Completeness gives a finite limit L. The next step must control *all* inputs near a, rather than only the selected sequence (t_n).

Fix $\epsilon > 0$ and choose a uniform tolerance $\delta > 0$ for output error $\epsilon/2$. Choose one index n with $t_n - a < \delta/2$ and $|f(t_n) - L| < \epsilon/2$. If $x \in (a, b)$ and $0 < x - a < \delta/2$, then $|x - t_n| \leq |x - a| + |t_n - a| < \delta$. Thus

$$|f(x) - L| \leq |f(x) - f(t_n)| + |f(t_n) - L| < \epsilon.$$

This proves the right-hand limit. The proof at b is analogous. □

Consequently a uniformly continuous function on a bounded open interval has finite endpoint limits and extends continuously to the closed interval by assigning those values. Conversely, a continuous extension to $[a, b]$ is uniformly continuous by Heine–Cantor, as is its restriction to (a, b). The extension is unique because its endpoint values must equal these one-sided limits.

Example 6.2 (Endpoint values and a bound). For $f(x) = \sqrt{x}$ on $(0, 1)$, the endpoint limits are 0 and 1, so the unique continuous extension is $F(x) = \sqrt{x}$ on $[0, 1]$. For $1/x$, the missing finite limit at zero rules out uniform continuity on $(0, 1)$.

More generally, every uniformly continuous function on a bounded open interval is bounded: extend it continuously to the closed interval and apply the extreme value theorem. The bounded function $\sin(1/x)$ from Example 5.4 shows that the converse fails.