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V. Let $A \in \mathbb{R}^{n \times m}$ have singular values $\sigma_1 \geq \cdots \geq \sigma_{\min\{n,m\}} \geq 0$.

By Theorem 2.9,

$$Av_i = \begin{cases} \sigma_i u_i & i=1,\dots,r \\ 0 & i=r+1,\dots,m \end{cases}$$

So, if $m > n$, then there is a v_i such that $Av_i = 0$. Hence, $\min_{\|x\|=1} \|Ax\| = 0$.

If $m \leq n$, then $Av_m = \sigma_m u_m$. Hence,

$$\|Av_m\| = \sigma_m \|u_m\| = \sigma_m$$

which implies that $\min_{\|x\|=1} \|Ax\| \leq \sigma_m$.

Now, let $x \in \mathbb{R}^m$ have unit length. Then, there are constants $c_1, \dots, c_m$ such that

$$x = c_1 v_1 + \cdots + c_m v_m$$

Therefore,

$$\begin{aligned} Ax &= c_1 Av_1 + \cdots + c_m Av_m \\ &= c_1 \sigma_1 u_1 + \cdots + c_m \sigma_m u_m \end{aligned}$$

which implies that

$$\begin{aligned} \|Ax\|_2^2 &= (c_1\sigma_1)^2 + \cdots + (c_m \sigma_m)^2 \\ &\geq \sigma_m^2 (c_1^2 + \cdots + c_m^2) \\ &= \sigma_m^2 \quad \Rightarrow \|Ax\|_2 \geq \sigma_m \end{aligned}$$

Hence, $\min_{\|x\|=1} \|Ax\| = \sigma_m$.

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VI. Let $A_\varepsilon = U\Sigma_\varepsilon V^T$ where $\Sigma_\varepsilon = \Sigma + \varepsilon I$

and I is the $n \times m$ identity matrix.

Since $\varepsilon > 0$, it follows that A_ε has full rank (all singular values are positive).

Moreover,

$$\|A - A_\varepsilon\|_2 = \|U\Sigma V^T - U\Sigma_\varepsilon V^T\|_2$$

$$= \|U(\Sigma - \Sigma_\varepsilon)V^T\|_2$$

$$= \|\Sigma - \Sigma_\varepsilon\|_2$$

$$= \|\varepsilon I\|_2 = \varepsilon.$$

VII. Consider the SVD of A:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 & & & \\ 1/2 & & & \\ 1/2 & & & \\ 1/2 & & & \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} [1]$$

where the last 3 colums of U are not important for solving the least squares problem

Then, the least squares solution is

$$y_1 = \frac{\langle b, u_1 \rangle}{\sigma_1} = \frac{(a/2 + 5/2 + 3/2 + 1/2)}{2} = \frac{9}{2}$$

$$x_1 = V y_1 = [1]\,\frac{9}{2} = \frac{9}{2}.$$