MATH-456: Homework 1 Solution

Due: 1/20/2023

Assignment

I. Consider the 5-point midpoint formula for the derivative of $f(x)$:

$$\phi_0(h) = \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h}.$$

Using the Taylor series centered at x, we find that

$$\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + O(h^6),\\
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + O(h^6),\\
f(x+2h) &= f(x) + 2hf'(x) + \frac{4h^2}{2}f''(x) + \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f^{(4)}(x) + \frac{32h^5}{5!}f^{(5)}(x) + O(h^6),\\
f(x-2h) &= f(x) - 2hf'(x) + \frac{4h^2}{2}f''(x) - \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f^{(4)}(x) - \frac{32h^5}{5!}f^{(5)}(x) + O(h^6)
\end{aligned}$$

Therefore, we have

$$8f(x+h) - 8f(x-h) = 16hf'(x) + \frac{16h^3}{3!}f'''(x) + \frac{16h^5}{5!}f^{(5)}(x) + O(h^7)$$

and

$$f(x-2h) - f(x+2h) = -4hf'(x) - \frac{16h^3}{3!}f'''(x) - \frac{64h^5}{5!}f^{(5)}(x) + O(h^7),$$

and it follows that

$$\phi_0(h) = f'(x) - \frac{4h^4}{5!}f^{(5)}(x) + O(h^6).$$

Thus,

$$f'(x) = \phi_0(h) + \frac{4h^4}{5!}f^{(5)}(x) + O(h^6),$$

which implies that $\phi_0(h)$ is a 4th order approximation of the first derivative $f'(x)$.

II. Note that

$$16\phi_0(h/2) - \phi_0(h) = 15f'(x) + O(h^6).$$

Therefore,

$$\phi_1(h) = \frac{16}{15}\phi_0(h/2) - \frac{1}{15}\phi_0(h)$$

is a 6th order approximation of the first derivative $f'(x)$. In Figure 1, the error of ϕ_1 in computing the derivative of $\sin x$ at $x = \pi/3$ is compared to several other numerical differention methods: ϕ_0, the centered difference, and forward difference methods. Note that if the order of the method is $p \geq 1$, then one expects the optimal h value to be around $O\left(\mu^{\frac{1}{p+1}}\right)$ with a corresponding error bound of $O\left(\mu^{\frac{p}{p+1}}\right)$, where $\mu = 2^{-53}$ is the unit-roundoff parameter in double-precision arithmetic. This expectation is readily verified in Figure 1, noting that the forward difference has order 1, the centered difference has order 2, ϕ_0 has order 4, and ϕ_1 has order 6.

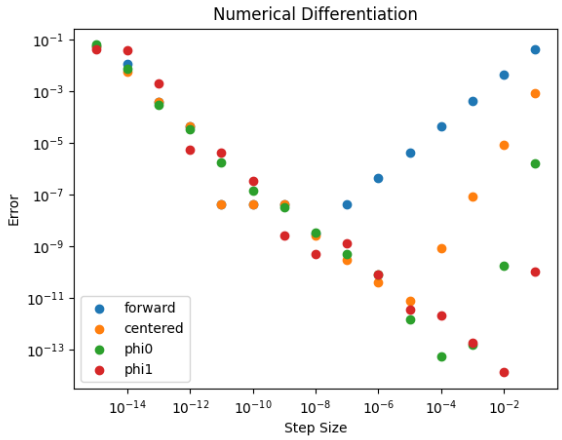

Figure 1: Numerical Differentiation Methods from Problem II.

III. a. Note that the differential equation can be written as

$$\frac{y'}{y} = \frac{t}{t^2+1}.$$

Integrating both sides gives us

$$\ln(y) = \frac{1}{2}\ln(t^2+1) + C,$$

hence, we have $y(t) = C\sqrt{t^2+1}$. The initial condition $y(0) = 1$ implies that $C = 1$. Therefore, the particular solution to the IVP is $y(t) = \sqrt{t^2+1}$. Let $M > 0$. Clearly, $f(t, y) = y\frac{t}{t^2+1}$ is continuous on the strip $[0, M] \times (-\infty, \infty)$. Furthermore, since $\frac{t}{t^2+1} <= 0.5$ for all $t \geq 0$, it follows that $f(t, y)$ is Lipschitz continuous in y. So, the IVP is well-posed and Theorem 1.6 implies that the above solution is unique on $[0, M] \times (-\infty, \infty)$.

b. Note that the differential equation can be written as

$$y' + 3y = 2t + e^{-t}.$$

Using the integration factor $\mu(t) = e^{3t}$, we have

$$\begin{aligned} y(t) &= e^{-3t}\int \left(2te^{3t} + e^{2t}\right) dt \\ &= e^{-3t}\left(\frac{2}{9}e^{3t}(3t-1) + \frac{1}{2}e^{2t} + C\right). \end{aligned}$$

The initial condition $y(0) = 1$ implies that $C = \frac{13}{18}$. Therefore, the particular solution to the IVP is given by

$$y(t) = \frac{2}{9}(3t-1) + \frac{1}{2}e^{-t} + \frac{13}{18}e^{-3t}.$$

Let $M > 0$. Clearly, $f(t, y) = 3t + e^{-t} - 3y$ is continuous on the strip $[0, M] \times (-\infty, \infty)$ and Lipschitz continuous in y. So, the IVP is well-posed and Theorem 1.6 implies that the above solution is unique on $[0, M] \times (-infty, \infty)$.