

Branch and Bound

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1 The Branch-and-Bound Method

In the previous section we studied the cutting plane method, where valid inequalities are added to the LP relaxation in order to eliminate fractional solutions. Another approach is the *branch-and-bound method*. Instead of strengthening the LP relaxation, this method partitions the feasible region into smaller subproblems and searches those subproblems systematically.

Suppose that the LP relaxation of an integer linear program has optimal solution $\hat{\mathbf{x}}^*$ and suppose that \hat{x}_k^* is fractional for some variable x_k . Since x_k must be an integer in any feasible solution of the ILP, every integer feasible solution satisfies one of the two conditions

$$x_k \leq \lfloor \hat{x}_k^* \rfloor \quad \text{or} \quad x_k \geq \lfloor \hat{x}_k^* \rfloor + 1.$$

Thus we may divide the problem into two new subproblems obtained by adding these constraints. The process of creating new subproblems by imposing additional constraints on a fractional variable is called *branching*. Each resulting subproblem is called a *node* in the branch-and-bound tree.

For each node we solve the LP relaxation of the corresponding subproblem. If the LP relaxation of a node has optimal value \hat{z} , then

$$z^* \leq \hat{z}$$

for every integer feasible solution in that node. Thus the LP relaxation provides an *upper bound* on the integer optimal value. If we find an integer feasible solution \mathbf{x} with objective value $z(\mathbf{x})$, then this provides a *lower bound* on the optimal value. The branch-and-bound method works by comparing these upper and lower bounds.

A node in the branch-and-bound tree may be eliminated from further consideration under certain conditions. A node is said to be *fathomed* if it does not need to be explored further. There are three situations in which a node may be fathomed.

- (1) The LP relaxation of the node is infeasible.
- (2) The optimal solution of the LP relaxation is integral.
- (3) The optimal value of the LP relaxation is no better than the best integer solution currently known.

In each case the node cannot contain a better integer solution, so it may be discarded.

2 A Geometric Example

Consider the integer linear program

$$\begin{aligned} & \text{maximize} && z = 10x_1 + 8x_2 \\ & \text{subject to} && 11x_1 + 7x_2 \leq 38, \\ & && 7x_1 + 9x_2 \leq 35, \\ & && x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

Its LP relaxation has optimal solution

$$\hat{\mathbf{x}}^* = \left(\frac{97}{50}, \frac{119}{50} \right)$$

with objective value

$$\hat{z}^* = \frac{961}{25} = 38.44.$$

Since $\hat{x}_1^* = 97/50$ is fractional, a natural first branch is along the variable x_1 . Because every integer feasible solution must satisfy either $x_1 \leq 1$ or $x_1 \geq 2$, we create the two subproblems

$$P_A : x_1 \geq 2 \quad \text{and} \quad P_B : x_1 \leq 1.$$

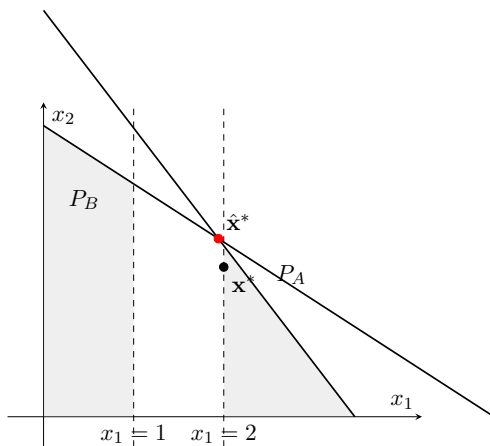


Figure 1: The first branch on x_1 splits the feasible region into the two subproblems P_A and P_B .

3 The First Two Nodes

The first two subproblems are

$$\begin{aligned} & \text{maximize} && z = 10x_1 + 8x_2 \\ & \text{subject to} && 11x_1 + 7x_2 \leq 38, \\ & && 7x_1 + 9x_2 \leq 35, \\ & && x_1 \geq 2, \\ & && x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

and

$$\begin{aligned} \text{maximize} \quad & z = 10x_1 + 8x_2 \\ \text{subject to} \quad & 11x_1 + 7x_2 \leq 38, \\ & 7x_1 + 9x_2 \leq 35, \\ & x_1 \leq 1, \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

The LP relaxation of P_A has optimal solution

$$\hat{\mathbf{x}}^* = \left(\frac{14}{7}, \frac{16}{7} \right) = \left(2, \frac{16}{7} \right)$$

with objective value

$$\hat{z}^* = \frac{268}{7}.$$

Since $\hat{x}_1^* = 2$ is integral but $\hat{x}_2^* = 16/7$ is not, we branch on x_2 and obtain the two new subproblems

$$P_{AA} : x_2 \geq 3 \quad \text{and} \quad P_{AB} : x_2 \leq 2.$$

The LP relaxation of P_B has optimal solution

$$\hat{\mathbf{x}}^* = \left(1, \frac{28}{9} \right)$$

with objective value

$$\hat{z}^* = \frac{314}{9}.$$

Again x_2 is fractional, so we branch on x_2 and obtain

$$P_{BA} : x_2 \geq 4 \quad \text{and} \quad P_{BB} : x_2 \leq 3.$$

4 The Remaining Nodes

The resulting subproblems are

$$P_{AA} : x_1 \geq 2, x_2 \geq 3,$$

$$P_{AB} : x_1 \geq 2, x_2 \leq 2,$$

$$P_{BA} : x_1 \leq 1, x_2 \geq 4,$$

$$P_{BB} : x_1 \leq 1, x_2 \leq 3.$$

Now P_{AA} is infeasible, so it is fathomed. Also, P_{BA} is infeasible, so it is fathomed.

The LP relaxation of P_{AB} has optimal solution

$$\hat{\mathbf{x}}^* = \left(\frac{24}{11}, \frac{22}{11} \right)$$

with objective value

$$\hat{z}^* = \frac{416}{11}.$$

Since this is still fractional, we branch again on x_1 , producing

$$P_{ABA} : x_1 \geq 3 \quad \text{and} \quad P_{ABB} : x_1 \leq 2.$$

The node P_{BB} has integral optimal solution

$$\mathbf{x}^* = (1, 3)$$

with objective value

$$z^* = 34.$$

Thus P_{BB} is fathomed and gives our first incumbent value 34.

The node P_{ABB} has integral optimal solution

$$\mathbf{x}^* = (2, 2)$$

with objective value

$$z^* = 36.$$

Hence P_{ABB} is fathomed and improves the incumbent to 36.

Finally, the LP relaxation of P_{ABA} has optimal value

$$\hat{z}^* = \frac{250}{7} < 36.$$

Therefore P_{ABA} is fathomed by bound, since it cannot contain an integer feasible solution better than the incumbent.

It follows that the optimal integer solution of the original problem is

$$\mathbf{x}^* = (2, 2), \quad z^* = 36.$$

5 The Branch-and-Bound Tree

The branching process is summarized by the following tree.

6 Summary

The branch-and-bound method proceeds as follows.

1. Solve the LP relaxation of the original ILP.
2. If the optimal solution is integral, stop.
3. Otherwise choose a fractional variable x_k and branch on it:

$$x_k \leq \lfloor \hat{x}_k^* \rfloor, \quad x_k \geq \lfloor \hat{x}_k^* \rfloor + 1.$$

4. Solve the LP relaxations of the resulting subproblems.
5. Fathom nodes whenever possible:
 - (a) if the node is infeasible,
 - (b) if the node has an integral optimal solution, or
 - (c) if its upper bound is no better than the incumbent.
6. Continue until all nodes are fathomed.

Remark. *Modern integer programming solvers combine branch and bound with cutting planes. This combined method is called branch-and-cut.*

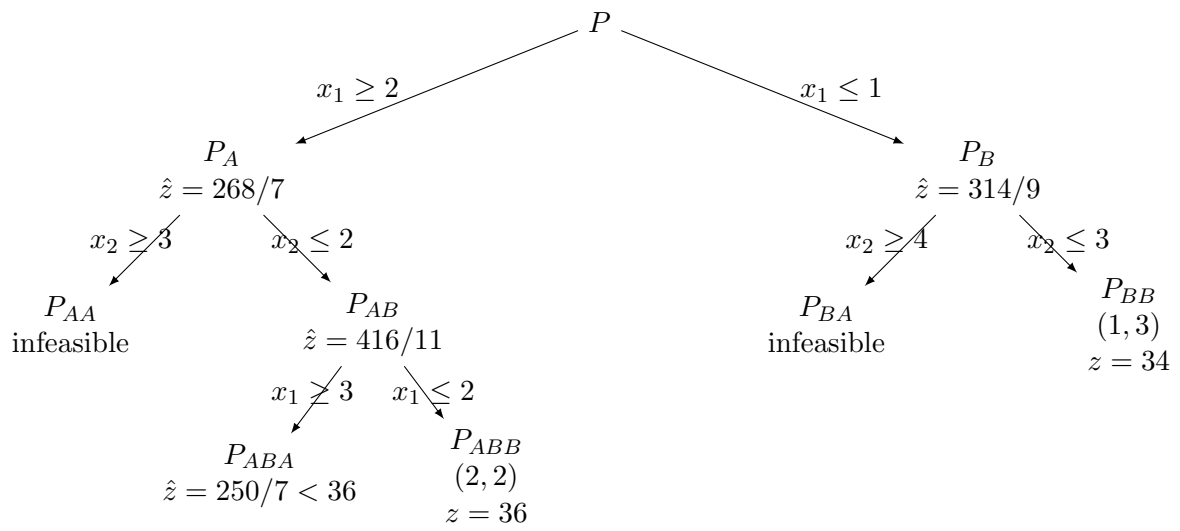


Figure 2: The branch-and-bound tree for the example problem.