Complementary Slackness Conditions

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1 Complementary Slackness

In the previous notes we established weak duality and used a worked example to motivate strong duality. In this section we describe *complementary slackness*, which gives a precise relationship between optimal primal solutions and optimal dual solutions. Complementary slackness provides an optimality test and explains how to read dual information directly from a final primal tableau (and vice versa).

1.1 Primal and dual slacks

Consider the primal–dual pair in standard form

$$\begin{aligned} \text{maximize} \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{i,j} x_j \leq b_i,\ 1 \leq i \leq m, \\ & x_j \geq 0,\ 1 \leq j \leq n \end{aligned} \tag{1}$$

and

$$\begin{aligned} \text{minimize} \quad & w = \sum_{i=1}^{m} b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{i,j} y_i \geq c_j,\ 1 \leq j \leq n, \\ & y_i \geq 0,\ 1 \leq i \leq m \end{aligned} \tag{2}$$

Definition 1. Let $\mathbf{x}$ be primal feasible and let $\mathbf{y}$ be dual feasible.

(a) The *primal slack vector* is $\mathbf{s} = \mathbf{b} - A\mathbf{x} \geq 0$.

(b) The *dual slack vector* is $\mathbf{t} = A^T\mathbf{y} - \mathbf{c} \geq 0$.

Thus, the ith primal constraint is *tight* if and only if $s_i = 0$, and the jth dual constraint is tight if and only if $t_j = 0$.

1.2 Complementary slackness

The following theorem provide a characterization of primal and dual optimal solutions.

Theorem 2 (Complementary Slackness)**.** *Let* $\mathbf{x}$ *be primal feasible and let* $\mathbf{y}$ *be dual feasible. Then* $\mathbf{x}$ *and* $\mathbf{y}$ *are optimal solutions to* (1) *and* (2) *if and only if*

$$y_i \left(b_i - \sum_{j=1}^{n} a_{i,j} x_j\right) = 0, \qquad 1 \le i \le m, \tag{3}$$

$$x_j \left(\sum_{i=1}^{m} a_{i,j} y_i - c_j\right) = 0, \qquad 1 \le j \le n. \tag{4}$$

Proof. Let $\mathbf{x}$ be primal feasible and $\mathbf{y}$ dual feasible. Since $\mathbf{s} = \mathbf{b} - A\mathbf{x} \ge 0$, $\mathbf{t} = A^T\mathbf{y} - \mathbf{c} \ge 0$, and $\mathbf{x}, \mathbf{y} \ge 0$, we have

$$\mathbf{y}^T\mathbf{s} \ge 0 \qquad \text{and} \qquad \mathbf{x}^T\mathbf{t} \ge 0.$$

Now expand the primal–dual gap:

$$\begin{aligned}
\mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} &= \mathbf{y}^T\mathbf{b} - \mathbf{y}^T A\mathbf{x} + \mathbf{y}^T A\mathbf{x} - \mathbf{c}^T\mathbf{x} \\
&= \mathbf{y}^T(\mathbf{b} - A\mathbf{x}) + \mathbf{x}^T(A^T\mathbf{y} - \mathbf{c}) \\
&= \mathbf{y}^T\mathbf{s} + \mathbf{x}^T\mathbf{t} \\
&= \sum_{i=1}^{m} y_i s_i + \sum_{j=1}^{n} x_j t_j.
\end{aligned}$$

Each term in the final expression is nonnegative.

($\Rightarrow$) If $\mathbf{x}$ and $\mathbf{y}$ are optimal, then by strong duality $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$, and hence the sum of nonnegative terms above is 0. Therefore, $y_i s_i = 0$ for each i and $x_j t_j = 0$ for each j, which is exactly (3)–(4).

($\Leftarrow$) If (3)–(4) hold, then $\mathbf{y}^T\mathbf{s} = 0$ and $\mathbf{x}^T\mathbf{t} = 0$, so $\mathbf{b}^T\mathbf{y} = \mathbf{c}^T\mathbf{x}$. By weak duality this implies $\mathbf{x}$ and $\mathbf{y}$ are optimal. □

1.3 Interpretation and an optimality test

Complementary slackness can be read as two "either–or" statements.

(1) For each primal constraint i: either the constraint is slack ($s_i > 0$) and then $y_i = 0$, *or* $y_i > 0$ and then the constraint must be tight ($s_i = 0$).

(2) For each primal variable j: either $x_j = 0$, *or* the corresponding dual constraint must be tight ($t_j = 0$).

Corollary 3 (Certificate of optimality)**.** *If* $\mathbf{x}$ *is primal feasible,* $\mathbf{y}$ *is dual feasible, and the complementary slackness conditions* (3)–(4) *hold, then* $\mathbf{x}$ *is primal optimal and* $\mathbf{y}$ *is dual optimal.*

Proof. This is the ($\Leftarrow$) direction in Theorem 2. □

1.4 Example (continued)

Consider the primal and dual problems:

$$\begin{aligned}\text{maximize} \quad & z = 22x_1 + 31x_2 + 29x_3 \\ \text{subject to} \quad & x_1 + 4x_2 + 6x_3 \le 73, \\ & 5x_1 - 2x_2 + 3x_3 \le 68, \\ & x_1, x_2, x_3 \ge 0.\end{aligned}$$

and

$$\begin{aligned}\text{minimize} \quad & w = 73y_1 + 68y_2 \\ \text{subject to} \quad & y_1 + 5y_2 \ge 22, \\ & 4y_1 - 2y_2 \ge 31, \\ & 6y_1 + 3y_2 \ge 29, \\ & y_1, y_2 \ge 0.\end{aligned}$$

From the final tableaus, the optimal solutions are

$$\mathbf{x}^* = \left(\frac{418}{22}, \frac{297}{22}, 0\right) = \left(19, \frac{27}{2}, 0\right), \qquad \mathbf{y}^* = \left(\frac{199}{22}, \frac{57}{22}\right),$$

and the optimal value is

$$z^* = w^* = \frac{18403}{22}.$$

Check complementary slackness. First compute the primal slack vector at $\mathbf{x}^*$:

$$\mathbf{s} = \mathbf{b} - A\mathbf{x}^* = \begin{pmatrix}73\\68\end{pmatrix} - \begin{pmatrix}1 & 4 & 6\\5 & -2 & 3\end{pmatrix}\begin{pmatrix}19\\27/2\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Thus, $s_1 = s_2 = 0$, so (3) holds.

Next compute the dual slack vector at $\mathbf{y}^*$:

$$\mathbf{t} = A^T\mathbf{y}^* - \mathbf{c} = \begin{pmatrix}1 & 5\\4 & -2\\6 & 3\end{pmatrix}\begin{pmatrix}199/22\\57/22\end{pmatrix} - \begin{pmatrix}22\\31\\29\end{pmatrix} = \begin{pmatrix}0\\0\\727/22\end{pmatrix}.$$

Therefore, $t_1 = t_2 = 0$ and $t_3 = 727/22 > 0$. Complementary slackness (4) requires $x_3^* t_3 = 0$, which holds since $x_3^* = 0$. Hence, the complementary slackness conditions hold and Corollary 3 certifies that $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal.

1.5 A tableau perspective

Complementary slackness explains the pattern observed between optimal primal and dual tableaus. At optimality, each basic variable has a positive value given by the right-hand side, each nonbasic variable is 0, and the corresponding dual slack records whether the matching inequality is tight. In particular, the dual slack t_j is positive exactly when the primal variable x_j is forced to be 0, and the primal slack s_i is positive exactly when the dual variable y_i is forced to be 0.