Convex sets, Polyhedra, and Extreme Point

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1 Extreme Points

We have seen that the feasible basic solutions in the simplex algorithm correspond to the extreme points of the feasible region. The following result proves this observation in general.

Theorem 1. *Let $S \subset \mathbb{R}^n$ denote the feasible region of a LP in standard form. There is a bijection between the extreme points of S and the feasible basic solutions of the LP.*

Proof. Let $\mathbf{x} = [\mathbf{x}_p|\mathbf{x}_s]$ denote a feasible basic solution, where $\mathbf{x}_p$ denotes the problem variables of the LP and $\mathbf{x}_s$ denotes the slack variables introduced in the simplex algorithm. Since $\mathbf{x}$ is feasible, it follows that all variables are non-negative and $\mathbf{x}_p$ satisfies all constraints of the LP; hence, $\mathbf{x}_p$ is in S. Furthermore, $\mathbf{x}_p$ must be an extreme point of S. Otherwise, at least one variable in $\mathbf{x}_p$ has a non-zero value that can be increased and remain feasible. This increase in a basic variable corresponds to a non-basic variable that can be decreased, which contradicts $\mathbf{x}$ being a basic solution.

Let $\mathbf{x}_p$ denote an extreme point of S. Then, $\mathbf{x}_p$ is the intersection of n hyperplanes corresponding to the constraints of the LP, including the non-negative constraints, where the inequalities are replaced by equalities. Each hyperplane corresponds to a non-basic variable that has been set to zero. The remaining basic variables have a value that can be determined from $\mathbf{x}_p$. Hence, $\mathbf{x}_p$ corresponds to a feasible basic solution. □

2 Convexity

A region $S \subset \mathbb{R}^n$ is convex if for any $\mathbf{u}, \mathbf{v} \in S$ the line segment

$$\mathbf{l}_t = t\mathbf{u} + (1-t)\mathbf{v}, \ 0 \le t \le 1,$$

is in S for all $t \in [0,1]$. The following result shows that every half-space is convex.

Proposition 2. *Let* $\mathbf{a} \in \mathbb{R}^n$ *and* $b \in \mathbb{R}$. *Then, the half-space*

$$H = \left\{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{a}^T\mathbf{x} \le b\right\}$$

is convex.

Proof. Let $\mathbf{u}, \mathbf{v} \in H$ and define the line segment

$$\mathbf{l}_t = t\mathbf{u} + (1-t)\mathbf{v}, \ 0 \le t \le 1.$$

Note that

$$\begin{aligned} \mathbf{a}^T\mathbf{l}_t &= t\mathbf{a}^T\mathbf{u} + (1-t)\mathbf{a}^T\mathbf{v} \\ &\le tb + (1-t)b = b, \end{aligned}$$

for all $t \in [0,1]$. □

Next, we show that the intersection of any two convex sets is a convex set.

Proposition 3. *Let* $S, T \subset \mathbb{R}^n$ *denote two convex sets. Then,* $R = S \cap T$ *is a convex set.*

Proof. Let $\mathbf{u}, \mathbf{v} \in R$ and define the line segment

$$\mathbf{l}_t = t\mathbf{u} + (1-t)\mathbf{v}, \ 0 \le t \le 1.$$

Note that $\mathbf{u}, \mathbf{v} \in S$ and $\mathbf{u}, \mathbf{v} \in T$. Since S and T are both convex sets, it follows that $\mathbf{l}_t \in S$ and $\mathbf{l}_t \in T$ for all $t \in [0,1]$. Therefore, $\mathbf{l}_t \in R$ for all $t \in [0,1]$. □

It is worth noting that the entire intersection is required to maintain convexity, see Figure 1. The previous two results imply that every polyhedron, intersection of finitely many half-spaces, is a convex set. We state this result formally in the following theorem.

Theorem 4. *Let* P *be a polyhedron. Then,* P *is a convex set.*

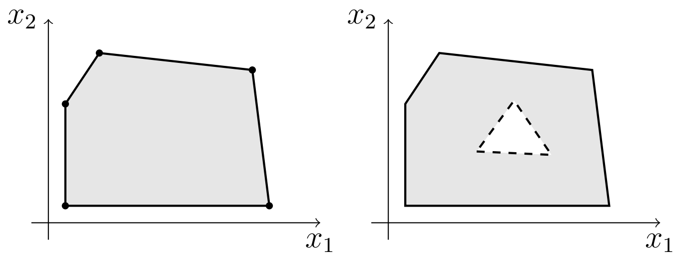

Figure 1: A convex set (left) versus a non-convex set formed by removing an interior region (right).

For a set of points $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^n$ and a collection of real numbers $c_1, \ldots, c_k$ we call the sum $\sum_{i=1}^k c_i \mathbf{x}_i$ a linear combination. If $c_1 + \cdots + c_k = 1$, then we call the sum an affine combination. If $c_i \geq 0$ for all $i \in \{1, \ldots, k\}$, then we call the sum a conic combination. If the sum is both affine and conic, then we say it is a convex combination.

Proposition 5. *Let $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^n$. Suppose that α and β are convex combinations of X and that γ is a convex combination of α and β. Then, γ is a convex combination of X.*

Proof. Note that

$$\begin{aligned}
&\alpha = a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k,\ a_i \geq 0,\ \sum_{i=1}^k a_i = 1,\\
&\beta = b_1\mathbf{x}_1 + \cdots + b_k\mathbf{x}_k,\ b_i \geq 0,\ \sum_{i=1}^k b_i = 1,\\
&\gamma = t\alpha + (1-t)\beta,\ t \in [0,1].
\end{aligned}$$

Therefore,

$$\gamma = (ta_1 + (1-t)b_1)\,\mathbf{x}_1 + \cdots + (ta_k + (1-t)b_k)\,\mathbf{x}_k,$$

where $(ta_i + (1-t)b_i) \geq 0$ and

$$\begin{aligned}
\sum_{i=1}^k (ta_i + (1-t)b_i) &= t\sum_{i=1}^k a_i + (1-t)\sum_{i=1}^k b_i\\
&= t(1) + (1-t)(1) = 1.
\end{aligned}$$

□

Let $X \subseteq \mathbb{R}^n$. The convex hull of X, denoted $\text{convHull}(X)$, is the smallest convex set that contains X. The convex span of X, denoted $\text{convSpan}(X)$, is the set of all convex combinations of finitely many points from X. Proposition 5 implies that $\text{convSpan}(X)$ is a convex set; hence, we have the following result.

Proposition 6. *Let $X \subseteq \mathbb{R}^n$. Then,* $\text{convHull}(X) \subseteq \text{convSpan}(X)$.

2.1 Class Exercises

Recall the linear program in (1a)–(1e), with feasible region plotted in Figure 2.

$$
\begin{aligned}
\text{maximize} \quad & z = x_1 + x_2 && \text{(1a)}\\
\text{subject to} \quad & 3x_1 + 5x_2 \leq 90, && \text{(1b)}\\
& 9x_1 + 5x_2 \leq 180, && \text{(1c)}\\
& x_2 \leq 15, && \text{(1d)}\\
& x_i \geq 0, \ \forall i \in \{1, 2\} && \text{(1e)}
\end{aligned}
$$

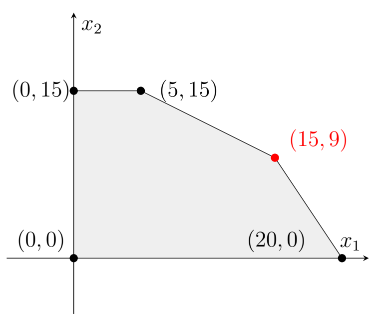

Figure 2: Feasible region for example LP in (1a)–(1e), with optimal solution in red.

I. The point $(10, 0)$ is feasible. What is the corresponding dictionary solution? How do you know this is not a basic solution?

II. The point $(10, 0)$ is feasible but not an extreme point. What is the only hyperplane that corresponds to this point?

III. The point $(20, 0)$ is feasible and an extreme point. What are the two hyperplanes that correspond to this point?