Cutting Planes

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1 The LP Relaxation

In this section we begin our study of integer linear programs. Recall that an *Integer Linear Program* (ILP) is a linear program together with the additional requirement that the decision variables take integer values. Thus the standard form of an ILP is as follows

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & x_i \in \mathbb{Z}_{\geq 0},\ \forall i \in \{1,\ldots,n\}, \end{aligned}$$

where $\mathbb{Z}_{\geq 0}$ denotes all non-negative integers.

If we drop the integrality requirement and allow $\mathbf{x} \geq 0$, then we obtain an ordinary linear program. The linear program obtained by deleting the integer constraints from an ILP is called the *LP relaxation* or *fractional relaxation* of the ILP. For a maximization ILP, the LP relaxation provides an upper bound on the integer optimal value. However, the optimal solution to the relaxation may be fractional, and therefore not feasible for the original integer problem.

2 Valid Inequalities and Cutting Planes

The central idea in the cutting plane method is to strengthen the LP relaxation by adding new linear constraints that are automatically satisfied by all integer feasible solutions. A linear inequality $\mathbf{a}^T\mathbf{x} \leq b$ is called a *valid inequality* for an ILP if every integer feasible solution satisfies the inequality. Furthermore, a valid inequality is called a *cutting plane* if it is violated by the current fractional optimal solution of the LP relaxation. Thus, a cutting plane removes the current fractional optimum without removing any integer feasible points.

A convenient way to produce valid inequalities is to take nonnegative linear combinations of the original constraints.

Theorem 1. *Consider the integer system*

$$A\mathbf{x} \leq \mathbf{b}, \qquad \mathbf{x} \in \mathbb{Z}^n_{\geq 0},$$

where A and $\mathbf{b}$ have integer entries. If $\mathbf{y} \geq 0$ and the row vector $\mathbf{y}^T A$ is integral, then

$$\left(\mathbf{y}^T A\right)\mathbf{x} \leq \left\lfloor \mathbf{y}^T\mathbf{b} \right\rfloor$$

is a valid inequality.

Proof. Let $\mathbf{x}$ be any integer feasible solution. Since $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{y} \geq 0$, we have

$$\left(\mathbf{y}^T A\right) \mathbf{x} \leq \mathbf{y}^T \mathbf{b}.$$

Because $\mathbf{y}^T A$ is integral and $\mathbf{x}$ is integral, the quantity $\left(\mathbf{y}^T A\right) \mathbf{x}$ is an integer. Therefore,

$$\left(\mathbf{y}^T A\right) \mathbf{x} \leq \left\lfloor \mathbf{y}^T \mathbf{b} \right\rfloor$$

Hence the inequality is valid for every integer feasible solution. □

3 A Geometric Example

Consider the integer linear program

$$\begin{aligned} \text{maximize} \quad & z = 10x_1 + 8x_2 \\ \text{subject to} \quad & 11x_1 + 7x_2 \leq 38, \\ & 7x_1 + 9x_2 \leq 35, \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

Its LP relaxation is obtained by replacing $x_1, x_2 \in \mathbb{Z}_{\geq 0}$ with $x_1, x_2 \geq 0$. The two constraint lines intersect at

$$\hat{\mathbf{x}}^* = \left(\frac{97}{50}, \frac{119}{50}\right),$$

see Figure 1. Note that $\hat{\mathbf{x}}^*$ is fractional and has an objective value

$$\hat{z}^* = 10\left(\frac{97}{50}\right) + 8\left(\frac{119}{50}\right) = \frac{961}{25} = 38.44.$$

A quick inspection of the integer feasible points, see Figure 1, shows that the integer optimum is

$$\mathbf{x}^* = (2, 2), \qquad z^* = 36.$$

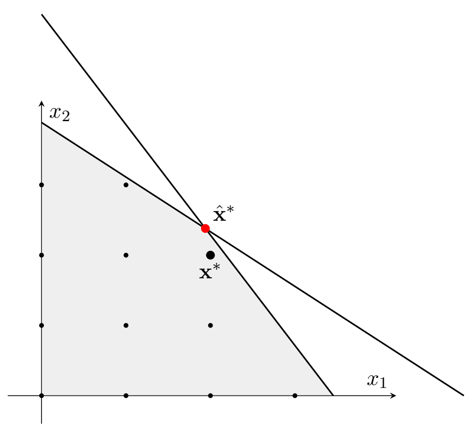

Figure 1: The LP relaxation of the integer program, with feasible region shaded, the fractional optimal solution identified in red, and the integer lattice points shown in black.

3.1 Two Simple Valid Inequalities

Using Theorem 1, we can derive valid inequalities directly from the original constraints.

First valid inequality

Take

$$\mathbf{y} = \frac{1}{10}\begin{bmatrix}3 & 1\end{bmatrix}^T.$$

Then

$$\mathbf{y}^T A = \frac{1}{10}\begin{bmatrix}3 & 1\end{bmatrix}\begin{bmatrix}11 & 7\\ 7 & 9\end{bmatrix} = \begin{bmatrix}4 & 3\end{bmatrix},$$

which is integral. Also,

$$\mathbf{y}^T\mathbf{b} = \frac{1}{10}(3 \cdot 38 + 1 \cdot 35) = \frac{149}{10}.$$

Therefore,

$$4x_1 + 3x_2 \le \left\lfloor \frac{149}{10} \right\rfloor = 14$$

is a valid inequality.

Second valid inequality

Take

$$\mathbf{y} = \frac{1}{25}\begin{bmatrix}1 & 2\end{bmatrix}^T.$$

Then,

$$\mathbf{y}^T A = \frac{1}{25}\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}11 & 7\\ 7 & 9\end{bmatrix} = \begin{bmatrix}1 & 1\end{bmatrix}$$

and

$$\mathbf{y}^T\mathbf{b} = \frac{1}{25}(38 + 70) = \frac{108}{25}.$$

Hence,

$$x_1 + x_2 \le \left\lfloor \frac{108}{25} \right\rfloor = 4$$

is also a valid inequality.

These two valid inequalities already force the integer optimum. Indeed, if we add

$$4x_1 + 3x_2 \le 14, \qquad x_1 + x_2 \le 4$$

to the model, then the multiplier vector $(0, 0, 2, 2)$ yields

$$10x_1 + 8x_2 = 2(4x_1 + 3x_2) + 2(x_1 + x_2) \le 2(14) + 2(4) = 36,$$

so $z^* \le 36$. Since $(2, 2)$ is feasible and gives $z = 36$, it is integer optimal.

3.2 Cutting Planes from a Fractional Tableau Row

The previous valid inequalities were obtained directly from multipliers on the original constraints. We now derive a cut from a fractional simplex row, which is the *Gomory fractional cut*. One may wonder why another cut generation method is needed. The answer is that the multiplier method requires insight into which multipliers to choose. In small examples this is easy, but in realistic integer programs there may be thousands of constraints, and the useful multiplier combinations are not obvious.

Matrix form of the LP relaxation

Introduce slack variables $x_3, x_4 \geq 0$ so that

$$\begin{aligned} 11x_1 + 7x_2 + x_3 &= 38, \\ 7x_1 + 9x_2 + x_4 &= 35. \end{aligned}$$

Then, the LP relaxation can be written in matrix form as

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x}_\pi + I\mathbf{x}_\beta = \mathbf{b}, \\ & \mathbf{x} \geq 0 \end{aligned}$$

where

$$A = \begin{bmatrix} 11 & 7 \\ 7 & 9 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 38 \\ 35 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}, \qquad \mathbf{x}_\pi = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{x}_\beta = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}.$$

At the optimal basis of the LP relaxation we have $\beta = \{1, 2\}$ and $\pi = \{3, 4\}$. Hence

$$B = \begin{bmatrix} 11 & 7 \\ 7 & 9 \end{bmatrix}, \qquad \Pi = I, \qquad \mathbf{c}_\beta = \begin{bmatrix} 10 \\ 8 \end{bmatrix}, \qquad \mathbf{c}_\pi = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The basic solution is

$$\mathbf{x}_\beta = B^{-1}b = \begin{bmatrix} 97/50 \\ 119/50 \end{bmatrix},$$

which is not integral. Also,

$$B^{-1} = \frac{1}{50}\begin{bmatrix} 9 & -7 \\ -7 & 11 \end{bmatrix},$$

so the dictionary is

$$\begin{aligned} x_1 &= \frac{97}{50} - \frac{9}{50}x_3 + \frac{7}{50}x_4, \\ x_2 &= \frac{119}{50} + \frac{7}{50}x_3 - \frac{11}{50}x_4. \end{aligned}$$

The objective row is

$$z = \frac{961}{25} - \frac{17}{25}x_3 - \frac{9}{25}x_4.$$

The first Gomory fractional cut

We use the first row of the dictionary:

$$x_1 + \frac{9}{50}x_3 - \frac{7}{50}x_4 = \frac{97}{50}.$$

Since this is an integer program, we are interested only in solutions for which all variables take integer values. The right-hand side is fractional, so this row can be used to derive a cut.

Theorem 2 (Gomory fractional cut)**.** *Suppose an integer program has a tableau row*

$$x_i + \sum_{j\in\pi} a_{ij}x_j = b_i,$$

where $i \in \beta$, $x_i, x_j \in \mathbb{Z}_{\geq 0}$ for all $j \in \pi$, and $b_i \notin \mathbb{Z}$. Then the inequality

$$\sum_{j\in\pi} (a_{ij} - \lfloor a_{ij} \rfloor)\, x_j \geq (b_i - \lfloor b_i \rfloor)$$

is valid for every integer feasible solution.

Proof. Write

$$a_{ij} = \lfloor a_{ij} \rfloor + f_{ij}, \qquad b_i = \lfloor b_i \rfloor + f_i,$$

where $0 \leq f_{ij} < 1$ and $0 < f_i < 1$. Then

$$x_i + \sum_{j\in\pi} \lfloor a_{ij} \rfloor x_j + \sum_{j\in\pi} f_{ij}x_j = \lfloor b_i \rfloor + f_i.$$

Rearranging,

$$\sum_{j\in\pi} f_{ij}x_j - f_i = \lfloor b_i \rfloor - x_i - \sum_{j\in\pi} \lfloor a_{ij} \rfloor x_j.$$

The right-hand side is an integer. Therefore the left-hand side is an integer. Since $\sum_{j\in\pi} f_{ij}x_j \geq 0$ and $0 < f_i < 1$, the only way for $\sum_{j\in\pi} f_{ij}x_j - f_i$ to be an integer is for it to be nonnegative. Hence

$$\sum_{j\in\pi} f_{ij}x_j \geq f_i,$$

which is the desired inequality. □

Applying Theorem 2 to the row

$$x_1 + \frac{9}{50}x_3 - \frac{7}{50}x_4 = \frac{97}{50},$$

we obtain

$$\frac{9}{50}x_3 + \frac{43}{50}x_4 \geq \frac{47}{50},$$

or equivalently,

$$9x_3 + 43x_4 \geq 47.$$

Now substitute

$$x_3 = 38 - 11x_1 - 7x_2, \qquad x_4 = 35 - 7x_1 - 9x_2.$$

Then

$$\begin{aligned} 9(38 - 11x_1 - 7x_2) + 43(35 - 7x_1 - 9x_2) &\geq 47, \\ 1847 - 400x_1 - 450x_2 &\geq 47, \\ 400x_1 + 450x_2 &\leq 1800. \end{aligned}$$

Dividing by 50 yields the valid inequality

$$8x_1 + 9x_2 \leq 36.$$

This is a cutting plane because it is satisfied by every integer feasible solution but is violated by the fractional optimum:

$$8\left(\frac{97}{50}\right) + 9\left(\frac{119}{50}\right) = \frac{1847}{50} > 36.$$

The second Gomory fractional cut

After adding the first Gomory fractional cut, our ILP becomes

$$\begin{aligned} \text{maximize} \quad & z = 10x_1 + 8x_2 \\ \text{subject to} \quad & 11x_1 + 7x_2 \le 38, \\ & 7x_1 + 9x_2 \le 35, \\ & 8x_1 + 9x_2 \le 36, \\ & x_1, x_2 \in \mathbb{Z}_{\ge 0} \end{aligned}$$

Introduce slack variables x_3, x_4, x_5 so that

$$\begin{aligned} 11x_1 + 7x_2 + x_3 &= 38, \\ 7x_1 + 9x_2 + x_4 &= 35, \\ 8x_1 + 9x_2 + x_5 &= 36. \end{aligned}$$

An optimal basis for the LP relaxation is $\beta = \{1, 2, 4\}$ and $\pi = \{3, 5\}$, which corresponds to

$$B = \begin{bmatrix} 11 & 7 & 0 \\ 7 & 9 & 1 \\ 8 & 9 & 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 38 \\ 35 \\ 36 \end{bmatrix}, \quad \mathbf{c}_\beta = \begin{bmatrix} 10 \\ 8 \\ 0 \end{bmatrix}, \quad \mathbf{c}_\pi = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The basic solution is given by

$$\mathbf{x}_\beta = B^{-1}\mathbf{b} = \frac{1}{43}\begin{bmatrix} 90 \\ 92 \\ 47 \end{bmatrix}.$$

The row of $\mathbf{x}_\beta = B^{-1}\left(\mathbf{b} - \Pi\mathbf{x}_\pi\right)$ corresponding to x_2 can be written as

$$x_2 - \frac{8}{43}x_3 + \frac{11}{43}x_5 = \frac{92}{43}.$$

Applying Theorem 2 gives us

$$35x_3 + 11x_5 \ge 6,$$

or (equivalently) in terms of problem variables we have

$$11x_1 + 8x_2 \le 40.$$

4 The Effect of Adding Cuts

We now illustrate the geometry of the cutting plane method. In each figure, the feasible region of the current LP relaxation is shaded, the current optimal relaxation solution is marked in red, and the newly added valid inequalities are shown as dashed lines, where each inequality is replaced by its corresponding equality.

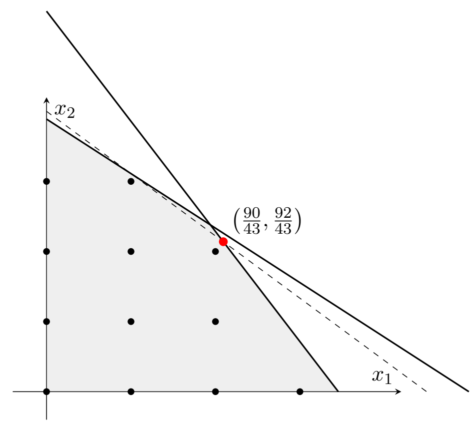

Figure 2: After adding the Gomory fractional cut $8x_1 + 9x_2 \leq 36$.

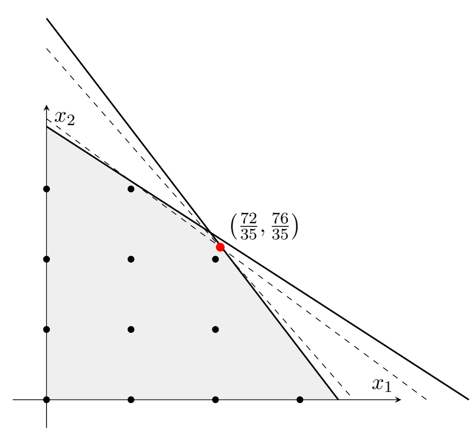

Figure 3: After adding second Gomory fractional cut $11x_1 + 8x_2 \leq 40$.