Homework 03 — Solutions

Math 482: Mathematical Methods of Operations Research (Spring 2026)
Week 6 (Feb 16–Feb 20, 2026)

Relevant topics: Primal–Dual Relationship, Strong Duality, Complementary Slackness.

Due: Friday February 20, 2026.

Instructions: Show your work clearly. When using the simplex method, clearly indicate your pivot choices (least subscript rule) and the corresponding basic feasible solutions.

I. Consider the following primal linear program.

$$\begin{aligned} \text{maximize} \quad & z = 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 4, \\ & 2x_1 + x_2 \le 5, \\ & x_1 + 3x_2 \le 7, \\ & x_i \ge 0, \ \forall i \in \{1, 2\} \end{aligned}$$

(a) **Solution.** Since (P) is a maximization problem in standard form ($\le$ constraints and $x \ge 0$), the dual is a minimization problem with $y \ge 0$ and $A^T y \ge c$:

$$\begin{aligned} \text{minimize} \quad & w = 4y_1 + 5y_2 + 7y_3 \\ \text{subject to} \quad & y_1 + 2y_2 + y_3 \ge 3, \\ & y_1 + y_2 + 3y_3 \ge 2, \\ & y_i \ge 0, \ \forall i \in \{1, 2, 3\} \end{aligned}$$

(b) **Solution.** The feasible region is a polygon in $\mathbb{R}^2$. The extreme points are

$$(0, 0), \quad \left(\tfrac{5}{2}, 0\right), \quad \left(0, \tfrac{7}{3}\right), \quad \left(\tfrac{8}{5}, \tfrac{9}{5}\right).$$

Evaluating $z = 3x_1 + 2x_2$ at these points gives

$$0, \quad \tfrac{15}{2}, \quad \tfrac{14}{3}, \quad \tfrac{42}{5}.$$

Hence an optimal primal solution is

$$x^* = \left(\tfrac{8}{5}, \tfrac{9}{5}\right), \qquad z^* = \tfrac{42}{5}.$$

(c) **Solution.** Compute the slack in each primal constraint at x^*:

$$\begin{aligned} 4 - (x_1^* + x_2^*) &= 4 - \tfrac{17}{5} = \tfrac{3}{5} > 0, \\ 5 - (2x_1^* + x_2^*) &= 5 - \tfrac{25}{5} = 0, \\ 7 - (x_1^* + 3x_2^*) &= 7 - \tfrac{35}{5} = 0. \end{aligned}$$

By complementary slackness, the corresponding dual variable must satisfy

$$\tfrac{3}{5} > 0 \implies y_1^* = 0.$$

Since $x_1^* > 0$ and $x_2^* > 0$, complementary slackness also forces the dual inequalities to be tight:

$$y_1^* + 2y_2^* + y_3^* = 3,$$
$$y_1^* + y_2^* + 3y_3^* = 2.$$

With $y_1^* = 0$, this becomes

$$2y_2^* + y_3^* = 3, \qquad y_2^* + 3y_3^* = 2.$$

Solving yields

$$y_2^* = \tfrac{7}{5}, \qquad y_3^* = \tfrac{1}{5}.$$

Thus

$$y^* = \left(0, \tfrac{7}{5}, \tfrac{1}{5}\right)$$

is dual feasible, and

$$w^* = 4y_1^* + 5y_2^* + 7y_3^* = 0 + 7 + \tfrac{7}{5} = \tfrac{42}{5} = z^*.$$

This verifies strong duality for this primal–dual pair.

II. Consider the primal linear program

$$\begin{aligned} \text{maximize} \quad & z = 6x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 8, \\ & 2x_1 + x_2 \le 10, \\ & x_i \ge 0, \ \forall i \in \{1,2\} \end{aligned}$$

(a) **Solution.** Add slack variables $x_3, x_4 \ge 0$:

$$x_1 + x_2 + x_3 = 8, \qquad 2x_1 + x_2 + x_4 = 10.$$

The initial tableau is

$$\begin{array}{cc|cc|c} 1 & 1 & 1 & 0 & 8 \\ 2 & 1 & 0 & 1 & 10 \\ \hline -6 & -4 & 0 & 0 & 0 \end{array}$$

(b) **Solution.** In the objective row, the most negative reduced cost is in the x_1-column, so x_1 enters. The minimum ratio test gives $10/2 < 8/1$, so the second row leaves (pivot entry 2).

After pivoting:

$$\begin{array}{cc|cc|c} 0 & \frac{1}{2} & 1 & -\frac{1}{2} & 3 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 5 \\ \hline 0 & -1 & 0 & 3 & 30 \end{array}$$

Now the only negative reduced cost is in the x_2-column, so x_2 enters. The minimum ratio test gives $3/(1/2) < 5/(1/2)$, so the first row leaves (pivot entry $1/2$).

After pivoting:

$$\begin{array}{cc|cc|c} 0 & 1 & 2 & -1 & 6 \\ 1 & 0 & -1 & 1 & 2 \\ \hline 0 & 0 & 2 & 2 & 36 \end{array}$$

All reduced costs are now nonnegative, so this tableau is optimal. Therefore,

$$(x_1^*, x_2^*) = (2, 6), \qquad z^* = 36.$$

(c) **Solution.** The dual of

$$\max\{6x_1 + 4x_2 : Ax \le b,\ x \ge 0\}, \quad A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix},\ b = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$$

is

$$\begin{aligned} \text{minimize} \quad & w = 8y_1 + 10y_2 \\ \text{subject to} \quad & y_1 + 2y_2 \ge 6, \\ & y_1 + y_2 \ge 4, \\ & y_i \ge 0,\ \forall i \in \{1, 2\} \end{aligned}$$

From the optimal tableau, the objective row coefficients under the slack-variable columns are 2 and 2, so we may read off

$$y^* = (2, 2).$$

Check dual feasibility:

$$y_1^* + 2y_2^* = 2 + 4 = 6, \qquad y_1^* + y_2^* = 2 + 2 = 4,$$

so y^* is dual feasible.

Next, verify complementary slackness. At $x^* = (2, 6)$ we have

$$8 - (x_1^* + x_2^*) = 0, \qquad 10 - (2x_1^* + x_2^*) = 0,$$

so both primal constraints are tight, consistent with $y_1^* > 0$ and $y_2^* > 0$. Also, since $x_1^* > 0$ and $x_2^* > 0$, both dual inequalities are tight, which we already checked.

Finally, the dual objective value is

$$w^* = 8 \cdot 2 + 10 \cdot 2 = 36 = z^*.$$