

Homework 04

Math 482: Mathematical Methods of Operations Research (Spring 2026)

Week of April 6–April 10, 2026

Relevant topics: Linear Ordering Problem, Quadratic Linear Ordering Problem, Quadratic Programming.

Due: Friday April 10, 2026.

Instructions: Show your work clearly. When writing optimization models, use appropriate notation and justify your conclusions. When analyzing quadratic programs, clearly identify the Hessian matrix, critical point, and the role of the feasible region.

I. Consider the weighted digraph on vertex set $V = \{1, 2, 3, 4\}$ with weight matrix

$$A = \begin{bmatrix} 0 & 4 & 1 & 3 \\ 2 & 0 & 5 & 2 \\ 3 & 1 & 0 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}.$$

The corresponding linear ordering problem is to find an acyclic tournament of maximum total weight.

- (a) Write the linear ordering problem as a binary integer linear program.
- (b) Compute the objective value for the rankings

$$(1, 2, 3, 4), \quad (2, 3, 1, 4), \quad (2, 1, 3, 4).$$

- (c) Rewrite the objective function in terms of a selected set of variables and use this to obtain an upper bound on the objective value.
- (d) Use parts (b)–(c) to determine an optimal ranking. State clearly whether the optimal ranking is unique.

II. Using the same matrix A from Problem I, define the dominance margins by

$$m_{ij} = a_{ij} - a_{ji}, \quad i \neq j,$$

and define the mixing terms by

$$b_{ij,jk} = m_{ik} - m_{jk},$$

with all other quadratic coefficients equal to zero.

- (a) Compute the margins m_{ij} for all $i \neq j$.
- (b) Compute the nonzero mixing terms $b_{ij,jk}$.
- (c) Write the quadratic linear ordering problem using the objective

$$\sum_{i \neq j} a_{ij} x_{ij} + \epsilon \sum_{i \neq j} \sum_{j \neq k} b_{ij,jk} x_{ij} x_{jk},$$

where $\epsilon > 0$.

- (d) Determine which rankings from Problem I are optimal for the linear ordering problem, and compute the quadratic contribution for each of those rankings.

- (e) Show that the quadratic linear ordering problem has a unique optimal ranking.
- (f) Explain why ϵ must be chosen sufficiently small.

III. Consider the quadratic linear ordering problem with binary variables x_{ij} and linearization variables $y_{ij,kl}$.

- (a) Write the full linearized integer linear programming formulation using $y_{ij,kl}$.
- (b) Prove that the linearization constraints enforce

$$y_{ij,kl} = x_{ij}x_{kl}$$

whenever all variables are binary.

- (c) Explain why the LP relaxation of the linearized model is weaker than the LP relaxation of the linear ordering problem.
- (d) Estimate the number of variables and constraints in the linearized model as a function of n .
- (e) Explain why this linearization makes the quadratic linear ordering problem substantially harder to solve than the linear ordering problem.

IV. Let $G = (V, E)$ be the path graph on vertices $V = \{1, 2, 3, 4\}$ with edge set

$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}.$$

Suppose each vertex i is assigned a value x_i , and we would like adjacent vertices to have similar values while remaining close to the target data

$$\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}.$$

Consider the quadratic program

$$\text{minimize} \quad \sum_{\{i,j\} \in E} (x_i - x_j)^2 + \sum_{i=1}^4 (x_i - b_i)^2$$

- (a) Write the objective function in the form

$$\frac{1}{2} \mathbf{x}^T Q \mathbf{x} + \mathbf{c}^T \mathbf{x} + d,$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

- (b) Compute the matrix Q and vector \mathbf{c} .
- (c) Show that Q is positive definite.

(d) Find the unique minimizer by solving

$$Q\mathbf{x} = -\mathbf{c}.$$

(e) Interpret the minimizer in terms of smoothing on the graph.

V. Consider the quadratic program

$$\begin{aligned} \text{minimize} \quad & z = x_1^2 + 2x_2^2 + x_1x_2 - 6x_1 - 4x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 3, \\ & x_i \geq 0, \forall i \in \{1, 2\} \end{aligned}$$

- (a) Compute the critical point of the objective function.
- (b) Determine whether the critical point is feasible.
- (c) If the critical point is feasible, prove that it is optimal. If the critical point is not feasible, explain how the level sets determine the optimal solution.
- (d) Find the optimal solution and objective value.
- (e) Explain how the eigenvalues of the Hessian matrix justify your conclusion.