The Simplex Method

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1 Infeasibility

When the given LP has an initial dictionary (or tableau) that is infeasible, we use the auxiliary method to move the simplex algorithm from Phase I to Phase II. In general, given a LP in standard form

$$\begin{aligned} \text{maximize} \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{i,j} x_j \leq b_i,\ 1 \leq i \leq m, \\ & x_j \geq 0,\ 1 \leq j \leq n \end{aligned}$$

the auxiliary problem is defined as follows

$$\begin{aligned} \text{maximize} \quad & v = -x_0 \\ \text{subject to} \quad & -x_0 + \sum_{j=1}^{n} a_{i,j} x_j \leq b_i,\ 1 \leq i \leq m, \\ & x_j \geq 0,\ 0 \leq j \leq n \end{aligned}$$

Theorem 1 describes an important relationship between the primal LP and the auxiliary LP.

Theorem 1. *Let P denote a primal LP in standard form and let Q denote the corresponding auxiliary LP. Then, P is feasible if and only if Q is optimal at $v^* = 0$.*

Proof. Suppose P is feasible and let $\mathbf{x} = [x_1, \ldots, x_n]$ denote a feasible solution of P. Then,

$$\sum_{j=1}^{n} a_{i,j} x_j \leq b_i,$$

for $1 \leq i \leq m$. Therefore, every constraint of Q is satisfied for $x_0 = 0$. Since the objective function of the auxiliary problem is to maximize $v = -x_0$, where $x_0 \geq 0$, it follows that this solution is Q optimal and $v^* = 0$.

Conversely, suppose that Q is optimal at $v^* = 0$. Then, $x_0 = 0$, which implies that every constraint of Q is of the form

$$\sum_{j=1}^{n} a_{i,j} x_j \leq b_i,$$

for $1 \leq i \leq m$. Therefore $\mathbf{x} = [x_1, \ldots, x_n]$ is a feasible solution of P. $\square$

From the proof of Theorem 1, we see that every Q-optimal solution at $v^* = 0$ can be used to construct a P-feasible solution. Moreover, Theorem 2 shows how to construct a feasible tableau for the auxiliary LOP.

Theorem 2. *Suppose that x_j is a (basic) slack variable for a given auxiliary LP with an infeasible tableau. Then, trading x_0 with x_j yields a feasible dictionary (or tableau) if and only if the value of x_j in the basic solution is most negative.*

Proof. Without loss of generality, we can assume that the coefficients for each slack variable in a tableau are 1. Therefore, a tableau is infeasible if and only if there exists a negative b_j, for some $1 \leq j \leq m$. When trading x_0 with x_j, where $n+1 \leq j \leq n+m$, we perform the following row operations

$$-b_{j-n} + b_k,$$

for all $1 \leq k \leq n$, where $k \neq j-n$. So, trading x_0 with x_j results in a feasible tableau if and only if b_{j-n} is the most negative value, that is, the value of x_j in the basic solution is most negative. □

For example, consider the following LP whose initial tableau is shown in Table 1.

$$\begin{aligned} \text{maximize} \quad & z = -90x_1 - 180x_2 - 15x_3 \\ \text{subject to} \quad & -3x_1 - 9x_2 \leq -1, \\ & -5x_1 - 5x_2 - x_3 \leq -1, \\ & x_i \geq 0, \ \forall i \in \{1, 2, 3\} \end{aligned} \tag{1}$$

-3	-9	0	1	0	0	-1
-5	-5	-1	0	1	0	-1
90	180	15	0	0	1	0

Table 1: Initial Tableau for LP in (1).

Note that the basic solution to the tableau in Table 1 is $\mathbf{x} = [0, 0, 0, -1, -1]$, which is not feasible. Hence, we are in Phase I of the Simplex Algorithm. The auxiliary LP is shown below with initial tableau shown in Table 2.

$$\begin{aligned} \text{maximize} \quad & v = -x_0 \\ \text{subject to} \quad & -x_0 - 3x_1 - 9x_2 \leq -1, \\ & -x_0 - 5x_1 - 5x_2 - x_3 \leq -1, \\ & x_i \geq 0, \ \forall i \in \{0, 1, 2, 3\} \end{aligned} \tag{2}$$

-1	-3	-9	0	1	0	0	-1
-1	-5	-5	-1	0	1	0	-1
1	0	0	0	0	0	1	0

Table 2: Initial Tableau for Auxiliary LP in (2).

Following Theorem 2, we select x_0 as the entering variable and column 0 as the pivot column, with pivot entry $a_{0,0} = -1$. Note that x_0 is trading with x_4. Now, we apply row operations to make all other entries in the pivot column zero. The resulting tableau is shown in Table 3.

1	3	9	0	-1	0	0	1
0	-2	4	-1	-1	1	0	0
0	-3	-9	0	1	0	1	-1

Table 3: Auxiliary tableau after trading x_0 with x_4.

Note that the tableau in Table 3 corresponds to basic variables $\beta^{(1)} = \{0, 5\}$ and non-basic variables $\pi^{(1)} = \{1, 2, 3, 4\}$. Hence, the basic solution is $\mathbf{x}^{(1)} = [1, 0, 0, 0, 0, 0]$, which is feasible. Therefore, we are now in Phase II of the simplex algorithm. Next, we select x_1 as the pivot column, with pivot entry $a_{0,1} = 3$. Note that x_1 is trading with x_0. Applying row operations gives us the tableau shown in Table 4.

1	3	9	0	-1	0	0	1
$\frac{2}{3}$	0	10	-1	$-\frac{5}{3}$	1	0	$\frac{2}{3}$
1	0	0	0	0	0	1	0

Table 4: Auxiliary tableau after trading x_1 with x_0.

Note that the tableau in Table 4 corresponds to basic variables $\beta^{(2)} = \{1, 5\}$ and non-basic variables $\pi^{(2)} = \{0, 2, 3, 4\}$. Hence, the basic solution is $\mathbf{x}^{(2)} = [0, \frac{1}{3}, 0, 0, 0, \frac{2}{3}]$. Moreover, this tableau is optimal with corresponding value $v^* = 0$. So, Theorem 1 implies that the primal LP in (1) is feasible. To recover a corresponding feasible tableau, we use the tableau in Table 4 by dropping the variable x_0, which has a value of 0. Moreover, since the objective function should not include any basic variables, we restate the objective variable in terms of the variables x_2, x_3, x_4.

$$\begin{aligned} z &= -90x_1 - 180x_2 - 15x_3 \\ &= -30 \cdot 3x_1 - 180x_2 - 15x_3 \\ &= -30\,(1 - 9x_2 + x_4) - 180x_2 - 15x_3 \\ &= -30 + 90x_2 - 15x_3 - 30x_4 \end{aligned}$$

Hence, we have the following feasible tableau

3	9	0	-1	0	0	1
0	10	-1	$-\frac{5}{3}$	1	0	$\frac{2}{3}$
0	-90	15	30	0	1	-30

Table 5: Feasible tableau for LP in (1).

Now that we are in Phase II of the simplex algorithm, we can proceed to identify the optimal tableau. To this end, let x_2 be the entering variable and column 2 the pivot column, with pivot entry $a_{2,2} = 10$. Then, applying row operations gives us

3	0	$\frac{9}{10}$	$\frac{1}{2}$	$-\frac{9}{10}$	0	$\frac{2}{5}$
0	10	-1	$-\frac{5}{3}$	1	0	$\frac{2}{3}$
0	0	6	15	9	1	-24

Table 6: Optimal tableau for LP in (1).