The Lagrangian

Thomas R. Cameron

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1 Introduction

Consider the QP in the following form

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \end{aligned}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric, $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$. In the previous lecture, we showed that at an optimal solution $\mathbf{x}^*$ satisfies

$$Q\mathbf{x}^* + \mathbf{c} \perp \text{nul}(A),$$

which implies

$$Q\mathbf{x}^* + \mathbf{c} \in \text{row}(A) = \text{col}(A^T).$$

Therefore, there exists $\mathbf{y}$ such that

$$Q\mathbf{x}^* + \mathbf{c} = A^T\mathbf{y}.$$

This is the first-order optimality condition. When combined with the feasibility condition $A\mathbf{x} = \mathbf{b}$, we get the KKT system which is writen as

$$\begin{aligned} Q\mathbf{x} + \mathbf{c} &= A^T\mathbf{y} \\ A\mathbf{x} &= \mathbf{b} \end{aligned}$$

In this lecture, we introduce the Lagrangian function which encodes the KKT system as its critical points. The Lagrangian will be useful in extending our framework to include inequality constraints, non-linear constraints, and duality theory. Moreover, the Lagrangian also encodes the second-order optimality conditions which when combined with the first-order conditions provide necessary and sufficient conditions for optimality.

2 The Lagrangian

Let $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x}$ denote the quadratic in the QP. The lagrangina function is defined by

$$L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) - \mathbf{y}^T (A\mathbf{x} - \mathbf{b}).$$

Note that if $A\mathbf{x} = \mathbf{b}$, then $L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x})$. Otherwise, the term $-\mathbf{y}^T (A\mathbf{x} - \mathbf{b})$ penalizes infeasibility. In this case, the $\mathbf{y}$ controls how strong constraint violations are penalized.

2.1 Recovering the KKT System

We compute the gradients of the Lagrangian.

$$\nabla_{\mathbf{x}} L = Q\mathbf{x} + \mathbf{c} - A^T\mathbf{y}$$

$$\nabla_{\mathbf{y}} L = -(A\mathbf{x} - \mathbf{b})$$

Setting both equal to zero gives:

$$Q\mathbf{x} + \mathbf{c} = A^T\mathbf{y}, \qquad A\mathbf{x} = \mathbf{b}.$$

Therefore, the KKT system is exactly the system of equations obtained by finding a critical point of the Lagrangian.

2.2 Existence and uniqueness

In the previous lecture we proved the following result

Theorem 2.1. *Suppose that Q is positive definite on $\mathrm{nul}(A)$, that is, $\mathbf{d}^T Q\mathbf{d} > 0$ for all $\mathbf{d} \in \mathrm{nul}(A)$. If $A\mathbf{x} = \mathbf{b}$ is feasible, then the QP has a unique minimizer.*

This theorem provides sufficient conditions for a unique optimal solution of the QP. Just as the KKT system, the Lagrangian encodes these conditions. Indeed, note that

$$\nabla^2_{xx} L(\mathbf{x}, \mathbf{y}) = Q.$$

Therefore, a feasible QP has a unique minimizer if

$$\mathbf{d}^T \nabla^2_{xx} L(\mathbf{x}, \mathbf{y})\mathbf{d} > 0$$

for all $\mathbf{d} \in \mathrm{nul}(A)$.

3 Example

Consider the QP with

$$Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}.$$

The Lagrangian

$$\begin{aligned} L(\mathbf{x}, \mathbf{y}) &= \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} - \mathbf{y}^T(A\mathbf{x} - \mathbf{b}) \\ &= \frac{1}{2}\left(x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + x_3^2\right) - 2x_1 - y_1(x_2 - 1) - y_2(x_1 + x_3) \end{aligned}$$

is a quadratic that encodes both the objective and constraints of the QP.

3.1 Critical Points

The critical points of the Lagrangian are given by the KKT system

$$\begin{aligned} \nabla_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}) &= Q\mathbf{x} + \mathbf{c} - A^T\mathbf{y} = 0, \\ \nabla_{\mathbf{y}} L(\mathbf{x}, \mathbf{y}) &= A\mathbf{x} - \mathbf{b} = 0. \end{aligned}$$

In this particular example, we have

$$\begin{aligned} x_1 - x_2 - 2 - y_2 &= 0, \\ -x_1 + 2x_2 - x_3 - y_1 &= 0, \\ -x_2 + x_3 - y_2 &= 0, \\ x_2 - 1 &= 0, \\ x_1 + x_3 &= 0. \end{aligned}$$

Note that $x_2 = 1$, $x_3 = -x_1$, and

$$x_1 - x_2 - 2 = -x_2 + x_3.$$

Therefore,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

3.2 Hessian

The Hessian (in $\mathbf{x}$) of the Lagrangian is given by

$$\nabla^2_{xx} L(\mathbf{x}, \mathbf{y}) = Q.$$

The second-order optimality conditions state that the critical point of the Lagrangian is the unique minimizer of the LOP if Q is positive definite on $\text{nul}(A)$. Note that

$$\text{nul}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right).$$

Therefore, every non-zero $\mathbf{d} \in \text{nul}(A)$ is of the form

$$\mathbf{d} = t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Furthermore, $\mathbf{d}^T Q \mathbf{d} = 2t^2 > 0$ for all $t \neq 0$. Hence, Q is positive definite on $\text{nul}(A)$.