The Linear Ordering Problem

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1 The Linear Ordering Problem

The linear ordering problem (LOP) is a classical combinatorial optimization problem that arises in applications involving ranking and aggregation, triangulation of economic input-output tables, chronological ordering, customer preference modeling. Given a weighted digraph, the goal is to choose a direction between every pair of vertices so that the resulting spanning tournament is acyclic and has maximum weight. Equivalently, if $A = [a_{ij}]$ is a nonnegative matrix, then the LOP seeks a simultaneous permutation of the rows and columns of A that maximizes the sum of the entries above the main diagonal.

Let $n \geq 2$ and define $[n] = \{1, 2, \ldots, n\}$. Let $a_{ij} \in \mathbb{R}_{\geq 0}$ for all $i, j \in [n]$. We interpret a_{ij} as the benefit of placing item i before item j. For each ordered pair $i \neq j$, let

$$x_{ij} = \begin{cases} 1 & \text{if item } i \text{ is placed before item } j, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the LOP can be modeled as follows.

$$\begin{aligned} \text{maximize} \quad & z = \sum_{i \neq j : i,j \in [n]} a_{ij} x_{ij} && \text{(1a)} \\ \text{subject to} \quad & x_{ij} + x_{ji} = 1, \qquad \forall i < j : i, j \in [n],, && \text{(1b)} \\ & x_{ij} + x_{jk} + x_{ki} \leq 2, \qquad \forall i < j,\ i < k,\ j \neq k : i, j, k \in [n],, && \text{(1c)} \\ & x_{ij} \in \{0, 1\}, \quad \forall i \neq j : i, j \in [n]. && \text{(1d)} \end{aligned}$$

The decision variables x_{ij} describe a subdigraph on vertex set $[n]$. Constraint (1b) ensures that for every pair i, j, exactly one of the edges (i, j) and (j, i) is selected. Therefore, the subdigraph is a spanning tournament. Constraint (1c) prevents a directed 3-cycle. Since every directed cycle contains a directed 3-cycle in a tournament, constraints (1b)–(1c) ensure that the subdigraph is acyclic. Hence, every feasible solution corresponds to a linear ordering of the vertices.

1.1 Digraph background

A *digraph* is a pair $\Gamma = (V, E)$ where V is a finite vertex set and $E \subseteq V \times V$ is a set of directed edges. A *tournament* is a digraph on V such that for every pair of distinct vertices $i, j \in V$, exactly one of (i, j) or (j, i) is present. An *acyclic digraph* is a digraph with no directed cycles. Thus, an acyclic tournament is exactly the same thing as a linear ordering of the vertices.

1.2 Why the constraints model a linear ordering

In this section, we provide a thorough argument for why constraints (1b)–(1c) model a linear ordering.

Tournament condition.

For each unordered pair $\{i, j\}$, constraint (1b) says

$$x_{ij} + x_{ji} = 1.$$

Since the variables are binary, exactly one of x_{ij} or x_{ji} equals 1. Therefore, exactly one directed edge is chosen between i and j.

Acyclicity condition.

Suppose that i, j, k are distinct vertices. If we had the directed cycle

$$i \to j, \qquad j \to k, \qquad k \to i,$$

then $x_{ij} = x_{jk} = x_{ki} = 1$, which would force

$$x_{ij} + x_{jk} + x_{ki} = 3,$$

contradicting (1c). Thus, every directed 3-cycle is forbidden.

Now assume a tournament contains some directed cycle. Choose a shortest directed cycle, say $v_1 \to v_2 \to \cdots \to v_t \to v_1$ with $t \geq 3$. Since the graph is a tournament, either $v_1 \to v_3$ or $v_3 \to v_1$. If $v_1 \to v_3$, then

$$v_1 \to v_3 \to v_4 \to \cdots \to v_t \to v_1$$

is a shorter directed cycle, a contradiction. Hence $v_3 \to v_1$, and then

$$v_1 \to v_2 \to v_3 \to v_1$$

is a directed 3-cycle, again a contradiction. Therefore, a tournament with no directed 3-cycle is acyclic.

It follows that every feasible solution to the LOP defines a total order

$$(v_1, v_2, \ldots, v_n).$$

1.3 Example: $n = 3$

Let

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 0 & 4 \\ 5 & 1 & 0 \end{bmatrix}.$$

Note that the objective function for the LOP can be written as

$$\begin{aligned} z &= 3x_{12} + 1x_{13} + 4x_{23} + 2x_{21} + 5x_{31} + 1x_{32} \\ &= 3x_{12} + 1x_{13} + 4x_{23} + 2(1 - x_{12}) + 5(1 - x_{13}) + 1(1 - x_{23}) \\ &= 8 + 1x_{12} - 4x_{13} + 3x_{23}. \end{aligned}$$

Hence, $z \leq 12$. However, $z = 12$ corresponds to the values $x_{12} = 1,\ x_{31} = 1,\ x_{23} = 1$, which is not feasible due to the cycle $1 \to 2 \to 3 \to 1$. If we switch $x_{12} = 0$ and $x_{21} = 1$, then we have a total order of $(2, 3, 1)$ corresponding to the optimal objective value of $z^* = 11$.

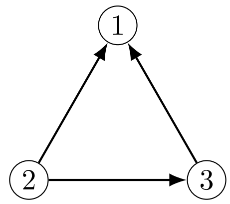

Figure 1: Graph interpretation of the optimal ordering $(2, 3, 1)$ for the $n = 3$ example.

Graph interpretation

The ordering $(2, 3, 1)$ corresponds to the acyclic tournament shown in Figure 1.

The linear ordering polytope for $n = 3$

For $n = 3$, the feasible solutions of the LOP correspond to the six permutations of $(1, 2, 3)$. Thus, the linear ordering polytope is the convex hull of these six 0–1 points in $\mathbb{R}^3$. Using the coordinates (x_{12}, x_{13}, x_{23}), the six feasible points are shown in Figure 2.

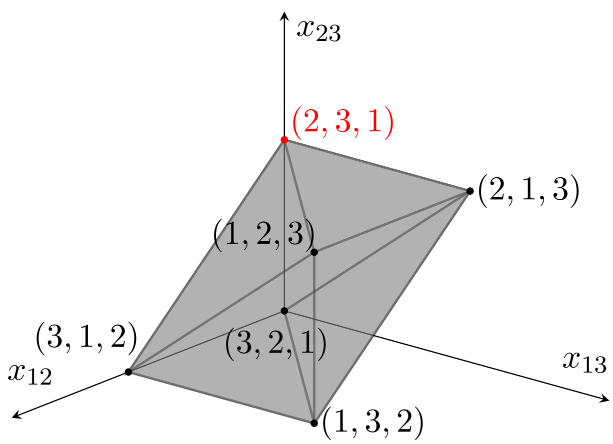

Figure 2: Linear Ordering Polytope: P^3_{LO} with optimal point in red for the $n = 3$ example.

1.4 Example: $n = 4$

Let

$$A = \begin{bmatrix} 0 & 2 & 6 & 4 \\ 7 & 0 & 5 & 3 \\ 1 & 4 & 0 & 8 \\ 2 & 6 & 1 & 0 \end{bmatrix}.$$

Note that the objective function for the LOP can be written as

$$\begin{aligned} z &= 2x_{12} + 6x_{13} + 4x_{14} + 5x_{23} + 3x_{24} + 8x_{34} \\ &\quad + 7(1 - x_{12}) + 1(1 - x_{13}) + 2(1 - x_{14}) + 4(1 - x_{23}) + 6(1 - x_{24}) + 1(1 - x_{34}) \\ &= 21 - 5x_{12} + 5x_{13} + 2x_{14} + 1x_{23} - 3x_{24} + 7x_{34}. \end{aligned}$$

Hence, $z \leq 36$.

Graph interpretation

Note that $z = 36$ is not feasible as shown in Figure 3, where the arcs shown correspond to $x_{ij} = 1$. The cycles $1 \to 4 \to 2 \to 1$, $2 \to 3 \to 4 \to 2$, and $1 \to 3 \to 4 \to 2 \to 1$ contradict the acyclic condition of the LOP.

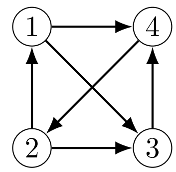

Figure 3: Graph interpretation of infeasible $z = 36$ for $n = 4$ example.

The three smallest terms of the objective function are $1x_{23}$, $2x_{14}$, and $-3x_{24}$. Note that changing the value of x_{23} and x_{14} will not remove all three dicycles in Figure 3; however, changing the value of x_{24} will remove all three dicycles. Thereofre, the optimal objective value is $z^* = 33$ and corresponds to the optimal ranking shown in Figure 4.

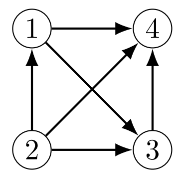

Figure 4: Graph interpretation of optimal $z^* = 36$, corresponding to ranking $(2, 1, 3, 4)$, for $n = 4$ example.

2 Integrality Gap of the Linear Ordering Problem

The linear ordering problems in Section 1.3 and Section 1.4 have an optimal relaxation value that is equal to the optimal LOP value. The gap between the optimal value of a relaxation and the optimal value of its LP is the *integrality gap*. It is know that the LOP has an integrality gap of 0 for $n \leq 5$.

Consider the digraph shown in Figure 5, where the edges shown have a weight of 1 while the edges not shown have a weight of 0. Hence, the LOP corresponding to this graph will have an objective function of

$$z = x_{14} + x_{25} + x_{36} + x_{42} + x_{43} + x_{51} + x_{53} + x_{61} + x_{62}.$$

Therefore, $z \leq 9$.

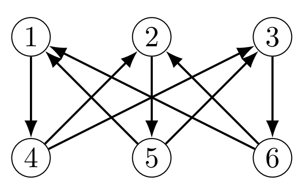

Figure 5: Graph instance for LOP with non-zero integrality gap.

Note that $z = 9$ is not feasible for the LOP with underlying graph in Figure 5. For example, $x_{14} = x_{42} = x_{25} = x_{51} = 1$ corresponds to the cycle $1 \to 4 \to 2 \to 5 \to 1$ which violates the acyclic condition of the LOP. In order to make a feasible solution, we must change the values of x_{14} and x_{36}, thereby not selecting those edges in the solution. We illustrate a feasible solution to the LOP in Figure 6, where the thick edges correspond to edges from the original digraph with a weight of 1 and the dashed edges correspond to edges from the original digraph with a weight of 0. Note that this feasible solution is optimal for the LOP, corresponds to the ranking $(4, 6, 2, 5, 3, 1)$ and objective value $z^* = 7$.

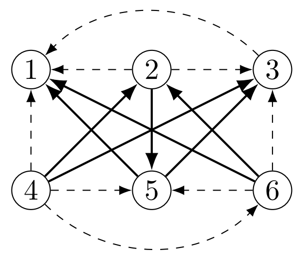

Figure 6: Feasible (optimal) solution for integral LOP induced by Graph in 5.

By relaxing the binary constraints on x_{ij} to $0 \leq x_{ij} \leq 1$ for all $i \neq j$, we loosen the restriction of the tournament and acyclic conditions of the LOP. Consider the optimal solution to the relaxed LOP in Figure 7, where all dashed edges are bidirectional and correspond to $x_{ij} = x_{ji} = 1/2$. Now the corresponding dicycles in the graph do not violate the tournament or acyclic conditions in the LOP. For example, $x_{12} = x_{21} = 1/2$ satisfies $x_{12} + x_{21} = 1$. Similarly, $x_{12} = 1/2$, $x_{25} = 1/2$, and $x_{51} = 1$ satisfy $x_{12} + x_{25} + x_{51} \leq 1$. Moreover, the corresponding objective value is

$$
\begin{aligned}
z &= x_{14} + x_{25} + x_{36} + x_{42} + x_{43} + x_{51} + x_{53} + x_{61} + x_{62} \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1 + 1 + 1 + 1 + 1 + 1 = 7.5,
\end{aligned}
$$

which is strictly greater than the objective value for the integral LOP.

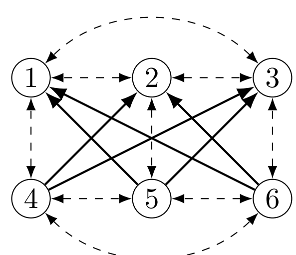

Figure 7: Optimal solution for relaxed LOP induced by Graph in 5.