Matrices and Dictionaries

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1 Matrix Form of the LP

When the simplex method is implemented in software like Gurobi, all information is stored in matrix form. Any linear program, in standard form, can be written in matrix form:

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq 0 \end{aligned}$$

where $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{x} \in \mathbb{R}^n$. We say that $\mathbf{x}$ is primal feasible if every constraint in the primal LP is satisfied. The dual LP can be written in the matrix form:

$$\begin{aligned} \text{minimize} \quad & w = \mathbf{b}^T\mathbf{y} \\ \text{subject to} \quad & A^T\mathbf{y} \geq \mathbf{c}, \\ & \mathbf{y} \geq 0 \end{aligned}$$

We say that $\mathbf{y}$ is dual feasible if every constraint in the dual LP is satisfied. Weak duality states that for any primal feasible $\mathbf{x}$ and dual feasible $\mathbf{y}$, we have

$$z = \mathbf{c}^T\mathbf{x} \leq \mathbf{y}^T A\mathbf{x} \leq \mathbf{y}^T\mathbf{b} = w.$$

For example, consider the following linear program.

$$\begin{aligned} \text{maximize} \quad & z = 46x_1 + 15x_2 + 12x_3 \\ \text{subject to} \quad & -7x_1 - x_2 - 3x_3 \leq -23, \\ & -2x_1 - 6x_2 - 8x_3 \leq -14, \\ & 4x_1 + 5x_2 + x_3 \leq 87, \\ & 9x_1 + 4x_2 + 3x_3 \leq 112, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Then, we have

$$A = \begin{bmatrix} -7 & -1 & -3 \\ -2 & -6 & -8 \\ 4 & 5 & 1 \\ 9 & 4 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -23 \\ -14 \\ 87 \\ 112 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 46 \\ 15 \\ 12 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

2 The Simplex Method in Matrix Form

The simplex method begins by introducing slack variables so that each constraint can be written as an equality, the resulting model is known as a dictionary. We can represent the dictionary in matrix form, but we need notation to differentiate between the basic and non-basic variables, which we denote by $\mathbf{x}_\beta$ and $\mathbf{x}_\pi$, respectively. Then, the primal LP has a dictionary in matrix form:

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T \mathbf{x}_\pi \\ \text{subject to} \quad & A\mathbf{x}_\pi + I\mathbf{x}_\beta = \mathbf{b}, \\ & \mathbf{x} \geq 0 \end{aligned}$$

However, the basis will change on each iteration of the simplex method, so we need a more general way to represent the dictionary. Given a basis β and parameter set π, define B and Π to be the columns of $[A|I]$ corresponding to β and π, respectively. Similarly, define $\mathbf{c}_\beta$ and $\mathbf{c}_\pi$ to be the entries of $[\mathbf{c}|\mathbf{0}]$ corresponding to β and π, respectively. Then, the primal LP has a dictionary in matrix form:

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}_\beta^T \mathbf{x}_\beta + \mathbf{c}_\pi^T \mathbf{x}_\pi \\ \text{subject to} \quad & B\mathbf{x}_\beta + \Pi\mathbf{x}_\pi = \mathbf{b}, \\ & \mathbf{x} \geq 0 \end{aligned}$$

Continuing our example, if $\beta = \{4, 5, 6, 7\}$ and $\pi = \{1, 2, 3\}$, then $B = I$, $\Pi = A$, $\mathbf{c}_\beta = 0$, $\mathbf{c}_\pi = \mathbf{c}$, and we have the initial dictionary. For a more interesting basis, let $\beta = \{2, 3, 5, 7\}$ and $\pi = \{1, 4, 6\}$. Then,

$$B = \begin{bmatrix} -1 & -3 & 0 & 0 \\ -6 & -8 & 1 & 0 \\ 5 & 1 & 0 & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix}, \quad \Pi = \begin{bmatrix} -7 & 1 & 0 \\ -2 & 0 & 0 \\ 4 & 0 & 1 \\ 9 & 0 & 0 \end{bmatrix}, \quad \mathbf{c}_\beta = \begin{bmatrix} 15 \\ 12 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_\pi = \begin{bmatrix} 46 \\ 0 \\ 0 \end{bmatrix}.$$

2.1 Basic Solution

Once the matrices B and Π have been identified, we can solve for $\mathbf{x}_\beta$ as follows:

$$\mathbf{x}_\beta = B^{-1} \left(\mathbf{b} - \Pi\mathbf{x}_\pi \right).$$

Furthermore, we can substitute these values into the objective function:

$$\begin{aligned} z &= \mathbf{c}_\beta^T B^{-1} \left(\mathbf{b} - \Pi\mathbf{x}_\pi \right) + \mathbf{c}_\pi^T \mathbf{x}_\pi \\ &= \mathbf{c}_\beta^T B^{-1} \mathbf{b} - \left(\mathbf{c}_\beta^T B^{-1} \Pi - \mathbf{c}_\pi^T \right) \mathbf{x}_\pi. \end{aligned}$$

Note that the basic solution can be formed by setting $\mathbf{x}_\pi = 0$. Then,

$$\mathbf{x}_\beta = B^{-1}\mathbf{b} \qquad z = \mathbf{c}_\beta^T B^{-1} \mathbf{b}.$$

To continue our example when $\beta = \{2, 3, 5, 7\}$ and $\pi = \{1, 4, 6\}$, note that we have a basic solution of

$$\mathbf{x}_\beta = \begin{bmatrix} 17 \\ 2 \\ 104 \\ 38 \end{bmatrix} \qquad z = 279,$$

which is feasible.

2.2 Improving the Basic Solution

If we allow the non-basic variables to be non-zero, then our objective function becomes

$$\begin{aligned} z &= \mathbf{c}_\beta^T B^{-1}\mathbf{b} - \left(\mathbf{c}_\beta^T B^{-1}\Pi - \mathbf{c}_\pi^T\right)\mathbf{x}_\pi \\ &= 279 - \begin{bmatrix} -\frac{197}{14} & -\frac{45}{14} & \frac{33}{14} \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \\ x_6 \end{bmatrix}. \end{aligned}$$

Hence, using the least subscript method, on the next iteration x_1 would enter the basis.

In general, suppose that $j \in \pi$ is going to enter the basis. Suppose that $x_j > 0$ and all other variables in the parameter set are zero, then the constraints in the LP can be written as

$$B\mathbf{x}_\beta + \Pi_j x_j = \mathbf{b},$$

where Π_j denotes the jth column of $[A|I]$ (column of Π corresponding to x_j). Therefore,

$$\mathbf{x}_\beta = B^{-1}\mathbf{b} - B^{-1}\Pi_j x_j.$$

In order for the basic solution to remain feasible, it follows

$$\left(B^{-1}\mathbf{b}\right)_i \geq \left(B^{-1}\Pi_j\right)_i x_j,$$

for all $1 \leq i \leq m$, where $(\cdot)_i$ denotes the ith entry of the given vector. If $\left(B^{-1}\Pi_j\right)_i \leq 0$, then this imposes no restriction. However, if $\left(B^{-1}\Pi_j\right)_i > 0$, then we require that

$$x_j \leq \frac{\left(B^{-1}\mathbf{b}\right)_i}{\left(B^{-1}\Pi_j\right)_i}.$$

The minimum ratio places the tightest restriction on the entering variable. Hence, we define the exiting variable to be the basic variable corresponding to

$$\underset{\left(B^{-1}\Pi_j\right)_i > 0}{\operatorname{argmin}} \frac{\left(B^{-1}\mathbf{b}\right)_i}{\left(B^{-1}\Pi_j\right)_i},$$

where we use the least subscript method to break ties.

Continuing with our example, our entering variable is x_1 and we have

$$B^{-1}\Pi_j = \frac{1}{14}\begin{bmatrix} 5 \\ 31 \\ 250 \\ 13 \end{bmatrix} \quad \text{and} \quad B^{-1}\mathbf{b} = \begin{bmatrix} 17 \\ 2 \\ 104 \\ 38 \end{bmatrix}.$$

Note that the minimum ratio is 28/31, which corresponds to the exiting variable x_3. Therefore, we now have a basis of $\beta = \{1, 2, 5, 7\}$ and parameter set of $\pi = \{3, 4, 6\}$. The corresponding dictionary matrices are shown below

$$B = \begin{bmatrix} -7 & -1 & 0 & 0 \\ -2 & -6 & 1 & 0 \\ 4 & 5 & 0 & 0 \\ 9 & 4 & 0 & 1 \end{bmatrix}, \ \Pi = \begin{bmatrix} -3 & 1 & 0 \\ -8 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}, \ \mathbf{c}_\beta = \begin{bmatrix} 46 \\ 15 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{c}_\pi = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}.$$

The basic solution is given by

$$\mathbf{x}_\beta = \frac{1}{31}\begin{bmatrix} 28 \\ 517 \\ 2724 \\ 1152 \end{bmatrix} \qquad z = \frac{9043}{31}.$$

If we allow the non-basic variables to be non-zero, then our objective function becomes

$$\begin{aligned} z &= \mathbf{c}_\beta^T B^{-1}\mathbf{b} - \left(\mathbf{c}_\beta^T B^{-1}\Pi - \mathbf{c}_\pi^T\right)\mathbf{x}_\pi \\ &= \frac{9043}{31} - \begin{bmatrix} \frac{197}{31} & -\frac{170}{31} & \frac{59}{31} \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_6 \end{bmatrix}. \end{aligned}$$

2.3 Dual Information

Let $\mathbf{y}^T = \mathbf{c}_\beta^T B^{-1}$. Then, $\mathbf{y}^T B = \mathbf{c}_\beta^T$ and it follows that

$$w = \mathbf{y}^T\mathbf{b} = \mathbf{c}_\beta^T B^{-1}\mathbf{b} = z.$$

However, $\mathbf{y}$ may not be feasible. Gurobi reports a measure of the dual infeasibility of $\mathbf{y}$, which takes into account whether the entries of $\mathbf{y}$ are non-negative and dual slacks are non-negative.

For example, when $\beta = \{2, 3, 5, 7\}$ and $\pi = \{1, 4, 6\}$, we have

$$B = \begin{bmatrix} -1 & -3 & 0 & 0 \\ -6 & -8 & 1 & 0 \\ 5 & 1 & 0 & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix}, \ \mathbf{c}_\beta = \begin{bmatrix} 15 \\ 12 \\ 0 \\ 0 \end{bmatrix}.$$

In this case, the dual vector is

$$\mathbf{y} = \frac{1}{14}\begin{bmatrix} -45 \\ 0 \\ 33 \\ 0 \end{bmatrix}.$$

Note that this vector is not dual feasible. In fact, $A^T\mathbf{y} \geq \mathbf{c}$ does not hold.

As another example, when $\beta = \{1, 2, 5, 7\}$ and $\pi = \{3, 4, 6\}$, we have

$$B = \begin{bmatrix} -7 & -1 & 0 & 0 \\ -2 & -6 & 1 & 0 \\ 4 & 5 & 0 & 0 \\ 9 & 4 & 0 & 1 \end{bmatrix}, \ \mathbf{c}_\beta = \begin{bmatrix} 46 \\ 15 \\ 0 \\ 0 \end{bmatrix}.$$

In this case, the dual vector is

$$\mathbf{y} = \frac{1}{31}\begin{bmatrix} -170 \\ 0 \\ 59 \\ 0 \end{bmatrix},$$

which is not dual feasible.

2.4 Why Must B Be Invertible?

Throughout, we have made the assumption that B must be invertible. If B were not an invertible, then the matrix equation

$$B\mathbf{x}_\beta = \mathbf{b}$$

either has no solution or infinitely many solutions. Suppose that we are in phase II of the simplex method. Then, the above system corresponds to a feasible basic solution. For LPs in standard form, we showed that there is a bijection between feasible basic solutions and extreme points of the feasible region. Therefore, there must be a unique solution. Thus, B is invertible.

3 Class Exercises

Complete the example problem to optimality. On each iteration, calculate the basic solution, objective value, reduced cost, corresponding dual vector, primal slacks, and dual slacks. On the final iteration, show that the dual vector is feasibl, the complementary slackness conditions hold, and conclude that the solution is optimal.