Primal–Dual Relationships

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1 Primal–Dual Relationships

Duality is the principle that every linear program has a companion problem whose feasible solutions provide *certificates* (upper bounds or lower bounds) on the objective value. In this section we derive the dual problem for a linear program in standard form and prove weak duality.

1.1 Standard form and the dual

We begin with a maximization problem in standard form.

$$\begin{aligned} \text{maximize} \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{i,j} x_j \leq b_i, \ 1 \leq i \leq m, \\ & x_j \geq 0, \ 1 \leq j \leq n \end{aligned} \tag{1}$$

Here $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$. A vector $\mathbf{x}$ satisfying the constraints is *primal feasible.*

To build an upper bound on $z = \mathbf{c}^T \mathbf{x}$, take a nonnegative linear combination of the constraints $A\mathbf{x} \leq \mathbf{b}$. Let $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y} \geq 0$. Then

$$A\mathbf{x} \leq \mathbf{b} \implies \mathbf{y}^T A \mathbf{x} \leq \mathbf{y}^T \mathbf{b}.$$

If we additionally enforce $A^T \mathbf{y} \geq \mathbf{c}$, then for every primal feasible $\mathbf{x}$,

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T A \mathbf{x} \leq \mathbf{y}^T \mathbf{b}.$$

Thus, any $\mathbf{y} \geq 0$ with $A^T \mathbf{y} \geq \mathbf{c}$ gives an upper bound on the primal objective. Minimizing that upper bound leads to the dual.

$$\begin{aligned} \text{minimize} \quad & w = \sum_{i=1}^{m} b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{i,j} y_i \geq c_j, \ 1 \leq j \leq n, \\ & y_i \geq 0, \ 1 \leq i \leq m \end{aligned} \tag{2}$$

A vector $\mathbf{y}$ satisfying the constraints is *dual feasible.*

1.2 Weak Duality

Theorem 1 (Weak Duality). *Let* $\mathbf{x}$ *be primal feasible and let* $\mathbf{y}$ *be dual feasible. Then,*

$$\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{y}.$$

In particular, if z^* *denotes the optimal primal value and* w^* *denotes the optimal dual value, then* $z^* \le w^*$.

Proof. Since $A^T\mathbf{y} \ge \mathbf{c}$ and $\mathbf{x} \ge 0$, we have $\mathbf{y}^T A\mathbf{x} \ge \mathbf{c}^T\mathbf{x}$. Since $A\mathbf{x} \le \mathbf{b}$ and $\mathbf{y} \ge 0$, we have $\mathbf{y}^T A\mathbf{x} \le \mathbf{y}^T\mathbf{b} = \mathbf{b}^T\mathbf{y}$. Combining these inequalities yields $\mathbf{c}^T\mathbf{x} \le \mathbf{b}^T\mathbf{y}$. □

Weak duality immediately explains three basic phenomena.

(a) If the primal is unbounded above, then the dual must be infeasible.

(b) If the dual is unbounded below, then the primal must be infeasible.

(c) If $\mathbf{x}$ and $\mathbf{y}$ are feasible and satisfy $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$, then $\mathbf{x}$ is primal optimal and $\mathbf{y}$ is dual optimal.

1.3 Example

Consider the primal linear program

$$\begin{aligned} \text{maximize} \quad & z = 22x_1 + 31x_2 + 29x_3 \\ \text{subject to} \quad & x_1 + 4x_2 + 6x_3 \le 73, \\ & 5x_1 - 2x_2 + 3x_3 \le 68, \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

Introduce slack variables $x_4, x_5 \ge 0$ so that

$$x_1 + 4x_2 + 6x_3 + x_4 = 73, \qquad 5x_1 - 2x_2 + 3x_3 + x_5 = 68.$$

Then, the initial tableau of the simplex method is shown in Table 1. Using the least subscript

1	4	6	1	0	0	73
5	-2	3	0	1	0	68
-22	-31	-29	0	0	1	0

Table 1: Initial tableau for the primal LP.

method, we arrive at the optimal tableau in Table 2.

0	22	27	5	-1	0	297
22	0	24	2	4	0	418
0	0	727	199	57	22	18403

Table 2: Optimal tableau for the primal LP.

Consider the dual linear program

$$\begin{aligned} \text{minimize} \quad & w = 73y_1 + 68y_2 \\ \text{subject to} \quad & y_1 + 5y_2 \geq 22, \\ & 4y_1 - 2y_2 \geq 31, \\ & 6y_1 + 3y_2 \geq 29, \\ & y_1, y_2 \geq 0. \end{aligned}$$

We transform the dual problem into standard form

$$\begin{aligned} \text{maximize} \quad & w = -73y_1 - 68y_2 \\ \text{subject to} \quad & -y_1 - 5y_2 \leq -22, \\ & -4y_1 + 2y_2 \leq -31, \\ & -6y_1 - 3y_2 \leq -29, \\ & y_1, y_2 \geq 0. \end{aligned}$$

Introduce slack variables $y_3, y_4, y_5 \geq 0$ so that

$$-y_1 - 5y_2 + y_3 = -22, \qquad -4y_1 + 2y_2 + y_4 = -31, \qquad -6y_1 - 3y_2 + y_5 = -29.$$

Then, the corresponding basic solution is

$$\mathbf{y} = [0, 0, -22, -31, -29],$$

which is infeasible. The corresponding auxiliary problem has the initial tableau shown in Table 3. Since y_4 has the most negative value in the infeasible basic solution, y_0 enters the basis and y_4 leaves

-1	-1	-5	1	0	0	0	-22
-1	-4	2	0	1	0	0	-31
-1	-6	-3	0	0	1	0	-29
1	0	0	0	0	0	1	0

Table 3: Auxiliary tableau for dual LP.

the basis, which results in the tableau shown in Table 4. The optimal tableau for the auxiliary

0	3	-7	1	-1	0	0	9
1	4	-2	0	-1	0	0	31
0	-2	-5	0	-1	1	0	2
0	-4	2	0	1	0	1	-31

Table 4: Feasible auxiliary tableau for dual LP.

problem is shown in Table 5. Since the auxiliary problem has an optimal solution of $v^* = 0$, we have a feasible dual LP. Note that the tableau in Table 5 has basis $\beta = \{y_1, y_2, y_5\}$ and non basic variables $\pi = \{y_0, y_3, y_4\}$. Therefore, the objective function of the dual problem (in standard form)

7	22	0	-2	-5	0	0	199
3	0	22	-4	1	0	0	57
29	0	0	-24	-27	22	0	727
1	0	0	0	0	0	1	0

Table 5: Optimal auxiliary tableau for dual LP.

is written as

$$\begin{aligned} w &= -73y_1 - 68y_2 \\ &= -73\left(\frac{199}{22} + \frac{2}{22}y_3 + \frac{5}{22}y_4\right) - 68\left(\frac{57}{22} + \frac{4}{22}y_3 - \frac{1}{22}y_4\right) \\ &= -\frac{18403}{22} - \frac{418}{22}y_3 - \frac{297}{22}y_4. \end{aligned}$$

In particular, a feasible tableau for the dual LP is shown in Table 6. Note that the tableau in

22	0	-2	-5	0	0	199
0	22	-4	1	0	0	57
0	0	-24	-27	22	0	727
0	0	418	297	0	22	-18403

Table 6: Feasible (optimal) tableau for dual LP.

Table 6 is also optimal since there are no negative coefficients in the objective row.

Note that the optimal tableaus for the primal and dual problems have a repetition of values. In particular, the coefficients of the objective row in Table 2 are the coefficients of the rightmost column in Table 6. Similarly, the coefficients of the objective row in Table 6 are the coefficients of the rightmost column in Table 2. More specifically, it appears that the problem values of $\mathbf{x}^*$ are the final coefficients of the dual slack variables, while the slack values of $\mathbf{x}^*$ are the final coefficients of the dual problem variables.

1.4 Strong Duality

Consider the standard form of the primal and dual problems shown in (1) and (2), respectively. Suppose that the simplex method halts with the tableau shown in Table 7.

c_1^*	$\cdots$	c_j^*	$\cdots$	c_n^*	c_{n+1}^*	$\cdots$	c_{n+i}^*	$\cdots$	c_{n+m}^*	1	z^*

Table 7: Optimal tableau for primal LP in (1).

Define

$$y_i^* = c_{n+i}^* \ (1 \le i \le m) \qquad y_{m+j}^* = c_j^* \ (1 \le j \le n).$$

Since the simplex has halted with the Table 7, it follows that the coefficients y_i^* $(1 \le i \le m)$ and y_{m+j}^* $(1 \le j \le n)$ are non-negative. Furthermore, the dictionary form of the primal LP implies that the following equation holds

$$z = z^* - \sum_{j=1}^{n} y_{m+j}^* x_j - \sum_{i=1}^{m} y_i^* x_{n+i},$$

for all problem variables $x_j \geq 0$ $(1 \leq j \leq n)$ and slack variables $x_{n+i} \geq 0$ $(1 \leq i \leq m)$. If we substitute the primal objective function and the definition of the slack variables, we obtain

$$\begin{aligned} \sum_{j=1}^{n} c_j x_j &= z^* - \sum_{j=1}^{n} y^*_{m+j} x_j - \sum_{i=1}^{m} y^*_i \left(b_i - \sum_{j=1}^{n} a_{i,j} x_j \right) \\ &= \left(z^* - \sum_{i=1}^{m} b_i y^*_i \right) - \sum_{j=1}^{n} y^*_{m+j} x_j + \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} x_j \right) y^*_i \\ &= \left(z^* - \sum_{i=1}^{m} b_i y^*_i \right) + \sum_{j=1}^{n} \left(\left(\sum_{i=1}^{m} a_{i,j} y^*_i \right) - y^*_{m+j} \right) x_j \end{aligned}$$

Since the above equation must hold for all possible problem variables, we set $x_j = 0$ $(1 \leq j \leq n)$ to obtain

$$z^* = \sum_{i=1}^{m} b_i y^*_i.$$

Therefore, we can rewrite the above equation as follows

$$\sum_{j=1}^{n} c_j x_j = \sum_{j=1}^{n} \left(\left(\sum_{i=1}^{m} a_{i,j} y^*_i \right) - y^*_{m+j} \right) x_j.$$

If we set $x_1 = 1$ and $x_j = 0$, for $2 \leq j \leq n$, we obtain

$$c_1 = \sum_{i=1}^{m} a_{i,1} y^*_i - y^*_{m+1}.$$

Since $y^*_{m+1} \geq 0$, it follows that $\sum_{i=1}^{m} a_{i,1} y^*_i \geq c_1$, which corresponds to the constraint $j = 1$ in the dual problem. Similarly, we can show that $\sum_{i=1}^{m} a_{i,j} y^*_i \geq c_j$ for each $1 \leq j \leq n$. Therefore, the values of y^*_i are dual feasible and optimal since $z^* = \sum_{i=1}^{m} b_i y^*_i$.

This result is known as strong duality, which we summarize in the following theorem.

Theorem 2 (Strong Duality)**.** *If the primal problem has an optimal solution z^*, then the dual problem has an optimal solution w^*; moreover, $z^* = w^*$.*