# The Quadratic Linear Ordering Problem

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## 1 Motivation

In the linear ordering problem (LOP), we see an acyclic tournamment subdigraph of maximum total edge weight. This assumes that pairwise comparisons contribute independently to the objective, ignoring higher-order consistency across multiple comparisons. For example, consider the following data, where the $(i, j)$ entry denotes how many times team $i$ beat team $j$:

$$A = \begin{bmatrix} 0 & 2 & 6 & 5 \\ 2 & 0 & 4 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The objective function for the corresponding LOP is given by

$$\begin{aligned} z &= 2x_{12} + 6x_{13} + 5x_{14} + 4x_{23} + 1x_{24} + 3x_{34} + 2(1 - x_{12}) \\ &= 2 + 6x_{13} + 5x_{14} + 4x_{23} + 1x_{24} + 3x_{34}. \end{aligned}$$

Hence, $z \leq 21$. Moreover, both rankings $(1, 2, 3, 4)$ and $(2, 1, 3, 4)$ are optimal with objective value $z = 21$. However, the ranking $(1, 2, 3, 4)$ is more consistent than the ranking $(2, 1, 3, 4)$ with respect to the data since team 1 won more games than team 2 against teams 3 and 4.

## 2 The Quadratic Linear Ordering Problem

Consider a digraph of order $n$, where each edge $(i, j)$ has weight $a_{ij} \geq 0$. In the LOP, the weight $a_{ij}$ defines the benefit of selecting $i$ to precede $j$ in the ranking. In the quadratic linear ordering problem, denoted QLOP, the weight $b_{ij,kl}$ defines the benefit of selecting $i$ to precede $j$ and $k$ to precede $l$. Defined below is the QLOP which seeks an acyclic tournament subdigraph that maximizes the total edge weight $a_{ij}$ plus the total mixing weight $b_{ij,kl}$.

$$\begin{aligned} \text{maximize} \quad & z = \sum_{i \neq j} a_{ij} x_{ij} + \sum_{i \neq j} \sum_{k \neq l} b_{ij,kl} x_{ij} x_{kl} \\ \text{subject to} \quad & x_{ij} + x_{ji} = 1, \qquad \forall i < j, \\ & x_{ij} + x_{jk} + x_{ki} \leq 2, \qquad \forall i < j,\ i < k,\ j \neq k, \\ & x_{ij} \in \{0, 1\}, \quad \forall i \neq j. \end{aligned}$$

**Example**

We now consider how the QLOP can be used to differentiate between the two optimal rankings in Section 1. For each edge $(i, j)$ define the dominance margin to be

$$m_{ij} = a_{ij} - a_{ji}$$

For each pair of edges $(i, j)$ and $(j, k)$ define the mixing term

$$b_{ij,jk} = m_{ik} - m_{jk},$$

and define all other mixing terms to be zero. We scale the non-zero mixing terms by a small $\epsilon > 0$ so that the objective seeks to optimize the total edge weight before optimizing the total mixing weight. For example,

$$\epsilon = \frac{\min\{a_{ij} : i \neq j,\ a_{ij} \neq 0\}}{2\max\{|b_{ij,jk}| : i \neq j,\ j \neq k\}},$$

will ensure that no quadratic term can outweigh a linear term in the objective. Then, the objective for the QLOP in this example can be written as follows

$$z = \sum_{i \neq j} a_{ij} x_{ij} + \epsilon \sum_{i \neq j} \sum_{j \neq k} b_{ij,jk} x_{ij} x_{jk}.$$

Since the objective function seeks to optimize the total edge weight before optimizing the total mixing weight, we only need to consider the quadratic terms on the two optimal rankings $(1, 2, 3, 4)$ and $(2, 1, 3, 4)$. For the ranking $(1, 2, 3, 4)$ the corresponding quadratic terms are

$$b_{12,23} = m_{13} - m_{23} = 6 - 4 = 2,$$
$$b_{12,24} = m_{14} - m_{24} = 5 - 1 = 4,$$
$$b_{13,34} = m_{14} - m_{34} = 5 - 3 = 2,$$
$$b_{23,34} = m_{24} - m_{34} = 1 - 3 = -2.$$

For the ranking $(2, 1, 3, 4)$ the corresponding quadratic terms are

$$b_{21,13} = m_{23} - m_{13} = 4 - 6 = -2,$$
$$b_{21,14} = m_{24} - m_{14} = 1 - 5 = -4,$$
$$b_{23,34} = m_{24} - m_{34} = 1 - 3 = -2,$$
$$b_{13,34} = m_{14} - m_{34} = 5 - 3 = 2.$$

Therefore, the ranking $(1, 2, 3, 4)$ has a quadratic sum of $6\epsilon$ and the ranking $(2, 1, 3, 4)$ has a quadratic sum of $-6\epsilon$. Since $\epsilon > 0$ it follows that the ranking $(1, 2, 3, 4)$ is the unique optimal ranking for the QLOP.

# 3 Linearization

Since the quadratic objective makes the problem difficult to solve directly, we introduce a standard linearization with the variables $y_{ij,kl} \in \{0, 1\}$ and constraints

$$y_{ij,kl} \leq x_{ij}, \tag{1}$$
$$y_{ij,kl} \leq x_{kl}, \tag{2}$$
$$y_{ij,kl} \geq x_{ij} + x_{kl} - 1. \tag{3}$$

Note that, if $x_{ij} = 0$ or $x_{kl} = 0$ then $y_{ij,kl} = 0$ by constraints (1)–(2). Conversely, if $y_{ij,kl} = 0$ then constraint (3) implies that $x_{ij} + x_{kl} \leq 1$, so $x_{ij} = 0$ or $x_{kl} = 0$. Also, if $x_{ij} = 1$ and $x_{kl} = 1$ then $y_{ij,kl} = 1$ by constraint (3). Conversely, if $y_{ij,kl} = 1$ then $x_{ij} = 1$ and $x_{kl} = 1$ by constraints (1)–(2).

Therefore, $y_{ij,kl} = x_{ij}x_{kl}$ when all variables are binary. Hence, the QLOP can be written as the following integer linear program.

$$
\begin{array}{lrl}
\text{maximize} & z = \displaystyle\sum_{i \neq j} a_{ij} x_{ij} + \sum_{i \neq j} \sum_{k \neq l} b_{ij,kl} y_{ij,kl} & \\
\text{subject to} & x_{ij} + x_{ji} = 1, & \forall i < j, \\
& x_{ij} + x_{jk} + x_{ki} \leq 2, & \forall i < j,\ i < k,\ j \neq k, \\
& y_{ij,kl} \leq x_{ij} & \forall i \neq j,\ k \neq l, \\
& y_{ij,kl} \leq x_{kl} & \forall i \neq j,\ k \neq l, \\
& y_{ij,kl} \geq x_{ij} + x_{kl} - 1 & \forall i \neq j,\ k \neq l, \\
& x_{ij} \in \{0,1\}, & \forall i \neq j, \\
& y_{ij,kl} \in \{0,1\}, & \forall i \neq j,\ k \neq l.
\end{array}
$$

Modeling the QLOP as a integer linear program allows us to solve using relaxation, cutting planes, and branch-and-bound methods, which utilize linear programming techniques and theory. However, this linear programming model comes with serious computational drawbacks.

- We have lost the polyhedral structure of the LOP, which has a tight relaxation and lots of known valid inequalities and cutting planes.
- The LP relaxation degrades significantly with the introduction of the variables $y_{ij,kl}$ since the relaxation of constraints (1)–(3) does not represent the product $x_{ij}x_{kl}$.
- The degraded relaxation makes it harder to design effective cuts or to guide branching.
- The number of variables increases from $O(n^2)$ to $O(n^4)$, which significantly increases the size of the model.