Introduction to Quadratic Programming

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1 Introduction

In the linear ordering problem (LOP), the objective function is linear in the decision variables. In the quadratic linear ordering problem (QLOP), we incorporate interactions between pairs of decisions. Recall that the QLOP objective is

$$\sum_{i\neq j} a_{ij}x_{ij} + \sum_{i\neq j}\sum_{k\neq l} b_{ij,kl}x_{ij}x_{kl},$$

where $x_{ij} \in \{0,1\}$ indicates whether i precedes j in the ordering. The quadratic terms capture interactions between decisions and lead naturally to *quadratic objective functions*. Quadratic models arise naturally in many applications, including:

- least squares approximation,
- portfolio optimization,
- energy minimization in mechanics,
- machine learning (e.g., support vector machines).

2 Quadratic Programming

The standard form of the quadratic program (QP) can be written as follows:

$$\begin{aligned} \text{maximize} \quad & z = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq 0 \end{aligned}$$

where $Q \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{m\times n}$, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^m$.

Without loss of generality, we may assume that Q is symmetric. Indeed, every matrix $Q \in \mathbb{R}^{n\times n}$ can be written as follows

$$Q = \frac{1}{2}\left(Q + Q^T\right) + \frac{1}{2}\left(Q - Q^T\right),$$

where $S(Q) = \frac{1}{2}\left(Q + Q^T\right)$ is the symmetric part of Q and $N(Q) = \frac{1}{2}\left(Q - Q^T\right)$ is the skew-symmetric part of Q. Note that

$$\mathbf{x}^T Q\mathbf{x} = \mathbf{x}^T S(Q)\mathbf{x} + \mathbf{x}^T N(Q)\mathbf{x} = \mathbf{x}^T S(Q)\mathbf{x}.$$

Next, we note that the factor of $\frac{1}{2}$ in the objective function is included to simplify the expression for the gradient. If we let $\mathbf{q}_j$ denote the jth column of Q, then

$$\begin{aligned}
\mathbf{x}^T Q \mathbf{x} &= \sum_{j=1}^{n} x_j \mathbf{x}^T \mathbf{q}_j \\
&= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{n} x_i q_{ij} \right) \\
&= \sum_{j=1}^{n} x_j^2 q_{jj} + \sum_{j=1}^{n} \sum_{i \neq j} x_i x_j q_{ij}.
\end{aligned}$$

If Q is symmetric, then $q_{ij} = q_{ji}$ for all $i \neq j$, so we may combine the off-diagonal terms to obtain

$$\mathbf{x}^T Q \mathbf{x} = \sum_{j=1}^{n} x_j^2 q_{jj} + 2 \sum_{1 \leq i < j \leq n} x_i x_j q_{ij}.$$

Therefore, the jth partial derivative of the quadratic is

$$\frac{\partial}{\partial x_j} \left(\frac{1}{2} \mathbf{x}^T Q \mathbf{x} \right) = x_j q_{jj} + \sum_{i \neq j} x_i q_{ij}$$

Furthermore, the gradient of the objective function is

$$\nabla(z) = Q\mathbf{x} + \mathbf{c}.$$

The critical point of the objective function is where the gradient is zero, that is, when

$$Q\mathbf{x} = -\mathbf{c}.$$

All unconstrained local extrema occur at critical points. In constrained problems, optimal solutions may also occur on the boundary of the feasible region.

3 Optimal Solutions

When solving a QP, we must take into account the shape of the quadratic $\frac{1}{2}\mathbf{x}^T Q \mathbf{x}$ and the feasible region $\{\mathbf{x} \colon A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq 0\}$. For example, if the quadratic is concave down then the critical point is an absolute maximum. Moreover, if the critical point is feasible then it is optimal for the maximization QP. Similarly, if the quadratic is concave up then the critical point is an absolute minimum. If the critical point is feasible then it is optimal for the minimization QP. Careful consideration must be given when the critical point is a saddle or when the critical point is not feasible.

Example

Consider the following QP:

$$\begin{aligned}
\text{minimize} \quad & z = x_1^2 + 2x_2^2 - 2x_1 - 4x_2 \\
\text{subject to} \quad & x_1 + x_2 \leq 5, \\
& x_1 \leq 4, \\
& x_1, x_2 \geq 0
\end{aligned}$$

The feasible region is shown in Figure 1. The matrices are given by

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}.$$

Hence, the critical point is at $(1, 1)$. Since the quadratic is concave up this critical point is an absolute minimum. Moreover, this point is feasible and is therefore optimal for the given QP.

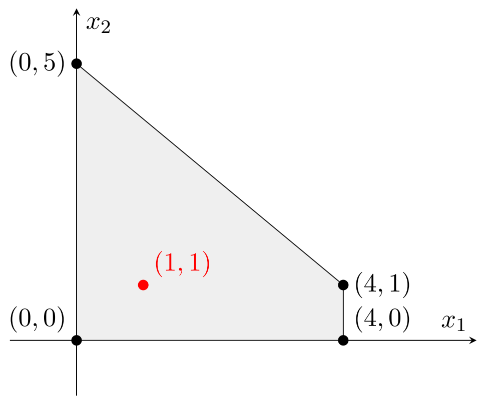

Figure 1: Feasible region of QP with optimal point in red.

Example

Consider the following QP:

$$\begin{aligned} \text{minimize} \quad & z = x_1^2 + x_1 x_2 + 2x_2^2 - 4x_1 - 6x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 2, \\ & x_2 \le \frac{3}{2}, \\ & x_1, x_2 \ge 0 \end{aligned}$$

The feasible region is shown in Figure 2. The matrices are given by

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}.$$

Solving the linear system $Q\mathbf{x} = -\mathbf{c}$ gives the critical point $(10/7, 8/7)$. Since the quadratic is concave up, this critical point is an absolute minimum. However, this critical point is not feasible. We increase the level value from $z(10/7, 8/7)$ until the corresponding level curve first intersects the feasible region. In Figure 2 we illustrate the level curve of $z = -6$ which is tangent to the feasible region at the point $(1, 1)$. Hence, the point $(1, 1)$ is optimal with corresponding objective value $z = -6$.

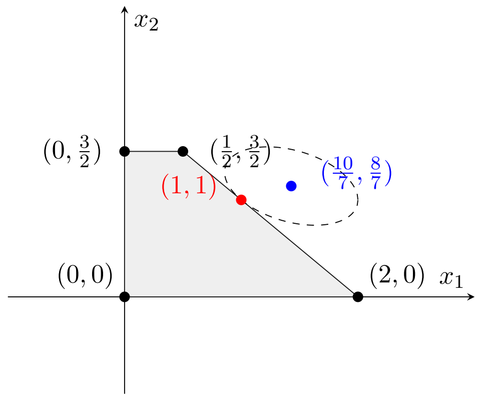

Figure 2: Feasible region of QP with optimal point in red, critical point in blue, and level curve dashed.

Example

Consider the following QP:

$$\begin{aligned} \text{minimize} \quad & z = x_1^2 - x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 3, \\ & x_2 \leq 2, \\ & x_1, x_2 \geq 0 \end{aligned}$$

The feasible region is shown in Figure 3. The matrices are given by

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the linear system $Q\mathbf{x} = -\mathbf{c}$ gives the critical point $(0, 0)$. Note that the objective function decreases as x_2 increases. Therefore, the optimal point is $(0, 2)$ with an objective value of $z = -4$.

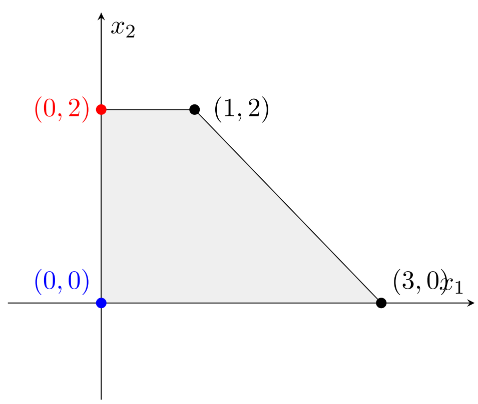

Figure 3: Feasible region of QP with optimal point in red and critical point in blue.

Example

Consider the following QP:

$$\begin{aligned} \text{minimize} \quad & z = -x_1^2 - x_2^2 + 4x_1 + 6x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 4, \\ & x_1 + 2x_2 \le 6, \\ & x_1, x_2 \ge 0 \end{aligned}$$

The feasible region is shown in Figure 4. The matrices are given by

$$Q = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

Solving the linear system $Q\mathbf{x} = -\mathbf{c}$ gives the critical point $(2, 3)$. Since the quadratic is concave down, this critical point is an absolute maximum. Since we are interested in minimizing the objective function, we want to decrease the objective value as much as possible while maintaining feasibility. In Figure 4 we will illustrate the level curve $z = 0$ whose objective value is as small as possible while still intersecting the feasible region. Both the points $(0, 0)$ and $(4, 0)$ are optimal.

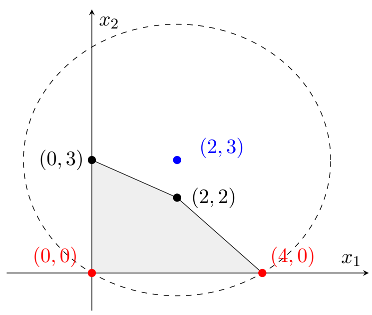

Figure 4: Feasible region of QP with optimal points in red, critical point in blue, and level curve dashed.