The Simplex Method

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1 Geometric Viewpoint

Let $S \subseteq \mathbb{R}^n$ be the feasible region of some LP and let $\mathbf{x} \in S$. We say that $\mathbf{x}$ is in the interior of S if there exists an $\epsilon > 0$ such that there is a ball of radius ϵ with center $\mathbf{x}$ that is contained in S; more precisely,

$$\{\mathbf{y} \in \mathbb{R}^n \colon \|\mathbf{x} - \mathbf{y}\| < \epsilon\} \subseteq S.$$

Otherwise, we say that $\mathbf{x}$ is on the boundary of S. A boundary point $\mathbf{x}$ is an extreme point of S if no line segment containing $\mathbf{x}$ in its interior has both its endpoints in S. Finally, we say that all infeasible points are exterior to S.

A feasible region S is bounded if it is contained in some ball. Otherwise, S is unbounded. In $\mathbb{R}^n$, every feasible region is a polyhedron since it is the intersection of finitely many half spaces. If the polyhedron is bounded, we call it a polytope. We say that F is a face of the polyhdron P if there is some half space $H = \left\{\mathbf{x} \colon \mathbf{a}^T\mathbf{x} \leq b\right\} \supseteq P$ such that $F = P \cap \partial H$, where $\partial H = \left\{\mathbf{x} \colon \mathbf{a}^T\mathbf{x} = b\right\}$. In this case, H is called the supporting hyperplane of F. A k-face of P is a face of dimension k. A facet is a face of dimension one less than the dimension of P and an extreme point is a face of dimension 0.

Before discussing the dimension of a polyhedron in further detail, note that every polyhedron has a finite number of extreme points since it is the intersection of finitely many half spaces. Furthermore, since the objective function is linear, if an LP has a non-empty bounded feasible region, then an optimal solution will occur at an extreme point. Therefore, we only need to consider the finite number of extreme points of the feasible region when looking for an optimal solution of an LP. However, the number of extreme points of the feasible region is typically exponential in the number of variables; so, even for a moderate size problem, we likely don't have time to check them all.

A polyhedron and all of its faces are an example of a convex set. The dimension of a convex set is defined as the dimension of its affine span, which is the set of all affine combinations of points from the set. Given the points $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$, an affine combination is defined by

$$\sum_{i=1}^{k} c_i \mathbf{x}_i,$$

where $\sum_{i=1}^{k} c_i = 1$. Note that affine combinations are linear combinations with an additional constraint. Furthermore, the points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are affinely independent if no point is an affine combination of the others. Equivalently, the points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are affinely independent if the points $\mathbf{x}_2 - \mathbf{x}_1, \ldots, \mathbf{x}_k - \mathbf{x}_1$ are linearly independent.

As an example, consider the points $\mathbf{x}_1 = (1, 0)$ and $\mathbf{x}_2 = (2, 0)$. These points are affinely independent since neither is an affine combination of the other. Equivalently, these points are affinely independent since the point $\mathbf{x}_2 - \mathbf{x}_1 = (1, 0)$ is linearly independent. The dimension of the affine span of $X = \{\mathbf{x}_1, \mathbf{x}_2\}$ is 1 since they both lie on the same line in $\mathbb{R}^2$; in particular, the affine hull of X is the entrie x-axis in $\mathbb{R}^2$.

1.1 Class Exercises

Recall the linear program in (1a)–(1e), with feasible region plotted in Figure 1.

$$\text{maximize} \quad z = x_1 + x_2 \tag{1a}$$
$$\text{subject to} \quad 3x_1 + 5x_2 \leq 90, \tag{1b}$$
$$9x_1 + 5x_2 \leq 180, \tag{1c}$$
$$x_2 \leq 15, \tag{1d}$$
$$x_i \geq 0, \ \forall i \in \{1, 2\} \tag{1e}$$

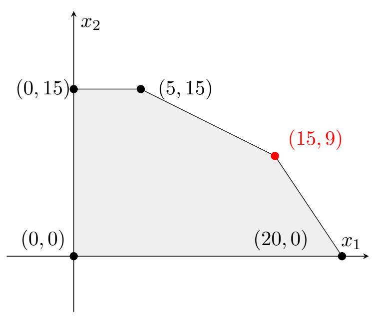

Figure 1: Feasible region for example LP in (1a)–(1e), with optimal solution in red.

I. Show that $F = \{(0, 0)\}$ is a face of the feasible region in Figure 1. What is the dimension of this face?

II. Show that $F = \{(x_1, x_2) \colon 0 \leq x_1 \leq 5, \ x_2 = 15\}$ is a face of the feasible region in Figure 1. What is the dimension of this face?

III. Show that the polytope in Figure 1 has dimension 2.