Simplex Phases and Advantages

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Recall the linear program from the matrix form notes.

$$\begin{aligned} \text{maximize} \quad & z = 46x_1 + 15x_2 + 12x_3 \\ \text{subject to} \quad & -7x_1 - x_2 - 3x_3 \leq -23, \\ & -2x_1 - 6x_2 - 8x_3 \leq -14, \\ & 4x_1 + 5x_2 + x_3 \leq 87, \\ & 9x_1 + 4x_2 + 3x_3 \leq 112, \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

After introducing slack variables x_4, x_5, x_6, x_7, the constraints of the linear program can be written as

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T \mathbf{x}_\pi \\ \text{subject to} \quad & A\mathbf{x}_\pi + I\mathbf{x}_\beta = \mathbf{b}, \\ & \mathbf{x} \geq 0 \end{aligned}$$

where

$$A = \begin{bmatrix} -7 & -1 & -3 \\ -2 & -6 & -8 \\ 4 & 5 & 1 \\ 9 & 4 & 3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -23 \\ -14 \\ 87 \\ 112 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 46 \\ 15 \\ 12 \end{bmatrix}, \ \mathbf{x}_\pi = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{x}_\beta = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}.$$

1 The Initial Basis

Since the basis can change, we introduce a more general form of the linear program

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}_\beta^T \mathbf{x}_\beta + \mathbf{c}_\pi^T \mathbf{x}_\pi \\ \text{subject to} \quad & B\mathbf{x}_\beta + \Pi\mathbf{x}_\pi = \mathbf{b}, \\ & \mathbf{x} \geq 0 \end{aligned}$$

where B and Π are formed from the columns of $[A|I]$ corresponding to β and Π, respectively. Similarly, $\mathbf{c}_\beta$ and $\mathbf{c}_\pi$ are the entries of $[\mathbf{c}|\mathbf{0}]$ corresponding to β and π, respectively.

Consider the initial basis $\beta = \{4, 5, 6, 7\}$ and $\pi = \{1, 2, 3\}$. Then, $B = I$ and $\Pi = A$. Moreover, the basic solution satisfies

$$Bx_\beta = b.$$

Since $B = I$, we have

$$x_\beta = \begin{bmatrix} -23 \\ -14 \\ 87 \\ 112 \end{bmatrix}.$$

Because x_4 and x_5 are negative, this basis is infeasible. Therefore, we are in Phase I.

2 Phase I in Matrix Form

We introduce x_0 and write the auxiliary program

$$\begin{aligned} \text{maximize} \quad & v = -x_0 \\ \text{subject to} \quad & B\mathbf{x}_\beta + \Pi\mathbf{x}_\pi - x_0\mathbf{1} = \mathbf{b}, \\ & \mathbf{x} \geq 0 \end{aligned}$$

where $\mathbf{1}$ is the all ones vector.

Since x_4 is the most negative entry in the infeasible basic solution, x_0 enters the basis and x_4 leaves the basis. Thus, we have $\beta = \{0, 5, 6, 7\}$ and $\pi = \{1, 2, 3, 4\}$. The matrices are

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad \Pi = \begin{bmatrix} -7 & -1 & -3 & 1 \\ -2 & -6 & -8 & 0 \\ 4 & 5 & 1 & 0 \\ 9 & 4 & 3 & 0 \end{bmatrix}, \quad \mathbf{c}_\beta = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_\pi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding basic solution is

$$\mathbf{x}_\beta = \begin{bmatrix} 23 \\ 9 \\ 110 \\ 135 \end{bmatrix} \qquad v = -23,$$

which is feasible.

If we allow the non-basic variables to be non-zero, then our objective function becomes

$$\begin{aligned} z &= \mathbf{c}_\beta^T B^{-1}\mathbf{b} - \left(\mathbf{c}_\beta^T B^{-1}\Pi - \mathbf{c}_\pi^T\right)\mathbf{x}_\Pi \\ &= -23 - \begin{bmatrix} -7 & -5 & -11 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \end{aligned}$$

The least subscript method states that x_1 will enter the basis. Furthermore, the exiting variable corresponds to the minimum ratio $(B^{-1}\mathbf{b})_i/(B^{-1}\Pi_1)_i$, where $(B^{-1}\Pi_1)_i > 0$. In this case, we have

$$B^{-1}\Pi_1 = \begin{bmatrix} 7 \\ 5 \\ 11 \\ 16 \end{bmatrix} \qquad B^{-1}\mathbf{b} = \begin{bmatrix} 23 \\ 9 \\ 110 \\ 135 \end{bmatrix}.$$

Therefore, the minimum ratio is 9/5, which corresponds to the exiting variable x_5.

We now have a basis of $\beta = \{0, 1, 6, 7\}$ and parameter set of $\pi = \{2, 3, 4, 5\}$. The matrices are

$$B = \begin{bmatrix} -1 & -7 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ -1 & 9 & 0 & 1 \end{bmatrix}, \ \Pi = \begin{bmatrix} -1 & -3 & 1 & 0 \\ -6 & -8 & 0 & 1 \\ 5 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 \end{bmatrix}, \ \mathbf{c}_\beta = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{c}_\pi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding basic solution is

$$\mathbf{x}_\beta = \frac{1}{5}\begin{bmatrix} 52 \\ 9 \\ 451 \\ 531 \end{bmatrix} \qquad v = -\frac{52}{5}.$$

The reduced cost is

$$\mathbf{c}_\beta^T B^{-1}\Pi - \mathbf{c}_\pi^T = \frac{1}{5}\begin{bmatrix} -40 & 5 & -85 & -105 \end{bmatrix}$$

The least subscript rule says that x_2 should enter the basis. The corresponding minimum ratio can be determined from the following vectors

$$B^{-1}\Pi_2 = \begin{bmatrix} 8 \\ -1 \\ 17 \\ 21 \end{bmatrix} \qquad B^{-1}\mathbf{b} = \begin{bmatrix} 52 \\ 9 \\ 451 \\ 531 \end{bmatrix}$$

Thus, the minimum ratio is $52/8$, which corresponds to the exiting variable x_0.

We now have a basis of $\beta = \{1, 2, 6, 7\}$ and parameter set of $\pi = \{0, 3, 4, 5\}$. The matrices are

$$B = \begin{bmatrix} -7 & -1 & 0 & 0 \\ -2 & -6 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ 9 & 4 & 0 & 1 \end{bmatrix}, \ \Pi = \begin{bmatrix} -1 & -3 & 1 & 0 \\ -1 & -8 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 \end{bmatrix}, \ \mathbf{c}_\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{c}_\pi = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding basic solution is

$$\mathbf{x}_\beta = \frac{1}{10}\begin{bmatrix} 31 \\ 13 \\ 681 \\ 789 \end{bmatrix} \qquad v = 0,$$

which is optimal at $v = 0$. Therefore, the primal LP has a feasible basis of $\beta = \{1, 2, 6, 7\}$.