The Simplex Method

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1 Unboundedness

We have seen that the auxiliary method can be used to identify a feasible basic solution, if they exist, or to show that the primal LP is infeasible. Another issue that the simplex method can detect is that of an unbounded LP. For example, consider the LP in (1).

$$\begin{aligned} \text{maximize} \quad & z = 3x_1 + 8x_2 \\ \text{subject to} \quad & -5x_1 + 2x_2 \leq 10, \\ & 2x_1 - 3x_2 \leq 6, \\ & x_i \geq 0, \ \forall i \in \{1, 2\} \end{aligned} \tag{1}$$

The initial tableau for the LP in (1) is shown in Table 1. Note that the corresponding basic variables are $\beta^{(0)} = \{3, 4\}$ and non-basic variables $\pi^{(0)} = \{1, 2\}$, with basic solution $\mathbf{x}^{(0)} = [0, 0, 10, 6]$ and $z^{(0)} = 0$.

-5	2	1	0	0	10
2	-3	0	1	0	6
-3	-8	0	0	1	0

Table 1: Initial tableau for the LP in (1).

We select x_1 as the entering variable with pivot entry $a_{2,1} = 2$. Applying row operations results in the tableau shown in Table 2. Note that the corresponding basic variables are $\beta^{(1)} = \{1, 3\}$ and non-basic variables $\pi^{(1)} = \{2, 4\}$, with basic solution $\mathbf{x}^{(1)} = [3, 0, 10, 0]$ and $z^{(1)} = 9$.

Note that the pivot row of the tableau in Table 2 indicates that we can increase value of z by increasing the value of x_2. However, there are no positive coefficients

0	$-\frac{11}{2}$	1	$\frac{5}{2}$	0	25
2	-3	0	1	0	6
0	$-\frac{25}{2}$	0	$\frac{3}{2}$	1	9

Table 2: Tableau for the LP in (1), after trading x_1 with x_4.

in the second column of the tableau. Therefore, trading x_2 with any basic variable results in a basic solution that is infeasible, which suggests that no variables place any restriction on the value of x_2; hence, the LP in (1) is unbounded.

2 Cycling

The last issue one might encounter when using the simplex algorithm is that of cycling. Note that there are only a finite number of possible tableau for any given LOP. Hence, if the simplex algorithm does not halt it must be due to cycling. The good news is that in 1977 Robert Bland proved that the use of the least subscript method ensures that the simplex algorithm will not cycle.

In what follows, we demonstrate that the simplex method can cycle when using the most negative subscript method. Consider the LP in (2).

$$\begin{aligned} \text{maximize} \quad & z = \frac{3}{4}x_1 - 150x_2 + \frac{1}{50}x_3 - 6x_4 \\ \text{subject to} \quad & \frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 \leq 0, \\ & \frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 \leq 0, \\ & x_3 \leq 1, \\ & x_i \geq 0, \quad \forall i \in \{1,2,3,4\} \end{aligned} \tag{2}$$

The initial tableau for the LP in (2) is shown in Table 3. Note that the corresponding basic variables are $\beta^{(0)} = \{5, 6, 7\}$ and non-basic variables $\pi^{(0)} = \{1, 2, 3, 4\}$, with basic solution $\mathbf{x}^{(0)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(0)} = 0$.

We select x_1 as the entering variable with pivot entry $a_{1,1} = 1/4$. Applying row operations results in the tableau shown in Table 4. Note that the corresponding basic variables are $\beta^{(1)} = \{1, 6, 7\}$ and non-basic variables $\pi^{(1)} = \{2, 3, 4, 5\}$, with basic solution $\mathbf{x}^{(1)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(1)} = 0$.

Next, we select x_2 as the entering variable with pivot entry $a_{2,2} = 30$. Applying row operations results in the tableau shown in Table 5. Note that the corresponding

$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0	0	0
$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0	0
0	0	1	0	0	0	1	0	1
$-\frac{3}{4}$	150	$-\frac{1}{50}$	6	0	0	0	1	0

Table 3: Initial tableau for the LP in (2).

1	-240	$-\frac{4}{25}$	36	4	0	0	0	0
0	30	$\frac{3}{50}$	-15	-2	1	0	0	0
0	0	1	0	0	0	1	0	1
0	-30	$-\frac{7}{50}$	33	3	0	0	1	0

Table 4: Tableau for the LP in (2), after trading x_1 with x_5.

basic variables are $\beta^{(2)} = \{1, 2, 7\}$ and non-basic variables $\pi^{(2)} = \{3, 4, 5, 6\}$, with basic solution $\mathbf{x}^{(2)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(2)} = 0$.

1	0	$\frac{8}{25}$	-84	-12	8	0	0	0
0	1	$\frac{1}{500}$	$-\frac{1}{2}$	$-\frac{1}{15}$	$\frac{1}{30}$	0	0	0
0	0	1	0	0	0	1	0	1
0	0	$-\frac{2}{25}$	18	1	1	0	1	0

Table 5: Tableau for the LP in (2), after trading x_2 with x_6.

Next, we select x_3 as the entering variable with pivot entry $a_{1,3} = 8/25$. Applying row operations results in the tableau shown in Table 6. Note that the corresponding basic variables are $\beta^{(3)} = \{2, 3, 7\}$ and non-basic variables $\pi^{(3)} = \{1, 4, 5, 6\}$, with basic solution $\mathbf{x}^{(3)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(3)} = 0$.

Next, we select x_4 as the entering variable with pivot entry $a_{2,4} = \frac{1}{40}$. Applying row operations results in the tableau shown in Table 7. Note that the corresponding basic variables are $\beta^{(4)} = \{3, 4, 7\}$ and non-basic variables $\pi^{(4)} = \{1, 2, 5, 6\}$, with basic solution $\mathbf{x}^{(4)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(4)} = 0$.

Next, we select x_5 as the entering variable with pivot entry $a_{1,5} = 50$. Applying row operations results in the tableau shown in Table 8. Note that the corresponding basic variables are $\beta^{(5)} = \{4, 5, 7\}$ and non-basic variables $\pi^{(5)} = \{1, 2, 3, 6\}$, with basic solution $\mathbf{x}^{(5)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(5)} = 0$.

Next, we select x_6 as the entering variable with pivot entry $a_{2,6} = 1/3$. Applying row operations results in the tableau shown in Table 9. Note that the corresponding basic variables are $\beta^{(6)} = \{5, 6, 7\}$ and non-basic variables $\pi^{(6)} = \{1, 2, 3, 4\}$, with basic solution $\mathbf{x}^{(6)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(6)} = 0$.

$\frac{25}{8}$	0	1	$-\frac{525}{2}$	$-\frac{75}{2}$	25	0	0	0
$-\frac{1}{160}$	1	0	$\frac{1}{40}$	$\frac{1}{120}$	$-\frac{1}{60}$	0	0	0
$-\frac{25}{8}$	0	0	$\frac{525}{2}$	$\frac{75}{2}$	-25	1	0	1
$\frac{1}{4}$	0	0	-3	-2	3	0	1	0

Table 6: Tableau for the LP in (2), after trading x_3 with x_1.

$-\frac{125}{2}$	10500	1	0	50	-150	0	0	0
$-\frac{1}{4}$	40	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0	0	0
$\frac{125}{2}$	-10500	0	0	-50	150	1	0	1
$-\frac{1}{2}$	120	0	0	-1	1	0	1	0

Table 7: Tableau for the LP in (2), after trading x_4 with x_2.

Before concluding this example, note that the least subscript method and the most negative method would choose the same pivot column until the tableau in Table 7. At this point, the least subscript method would select x_1 as the entering variable, with pivot entry $a_{3,1} = 125/2$, and corresponding leaving variable x_7. Applying row operations would result in the tableau shown in Table 10. Note that the corresponding basic variables are $\beta = \{1, 3, 4\}$ and non-basic variables $\pi = \{2, 5, 6, 7\}$, with basic solution $\mathbf{x} = [2/125, 0, 1, 1/250, 0, 0, 0]$ and $z = 1/125$. Hence, using the least subscript method we are able to break out of that cycle.

3 The Fundamental Theorem of Linear Programming

Now, we are ready to state the fundamental theorem of linear optimization.

Theorem 1. *Let P be an LOP in standard form. Then,*

a. P is either infeasible, unbounded, or it has a maximum.

b. If P has a feasible solution, then it has a feasible tableau.

c. If P has an optimal solution, then it has an optimal tableau.

$-\frac{5}{4}$	210	$\frac{1}{50}$	0	1	-3	0	0	0
$\frac{1}{6}$	-30	$-\frac{1}{150}$	1	0	$\frac{1}{3}$	0	0	0
0	0	1	0	0	0	1	0	1
$-\frac{7}{4}$	330	$\frac{1}{50}$	0	0	-2	0	1	0

Table 8: Tableau for the LP in (2), after trading x_5 with x_3.

$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0	0	0
$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0	0
0	0	1	0	0	0	1	0	1
$-\frac{3}{4}$	150	$-\frac{1}{50}$	6	0	0	0	1	0

Table 9: Tableau for the LP in (2), after trading x_6 with x_4.

0	0	1	0	0	0	1	0	1
0	-2	0	1	$\frac{2}{15}$	$-\frac{1}{15}$	$\frac{1}{250}$	0	$\frac{1}{250}$
1	-168	0	0	$-\frac{4}{5}$	$\frac{12}{5}$	$\frac{2}{125}$	0	$\frac{2}{125}$
0	36	0	0	$-\frac{7}{5}$	$\frac{11}{5}$	$\frac{1}{125}$	1	$\frac{1}{125}$

Table 10: Tableau obtained from Table 7 using least subscript rule.