Math 482 Workshop — Solutions

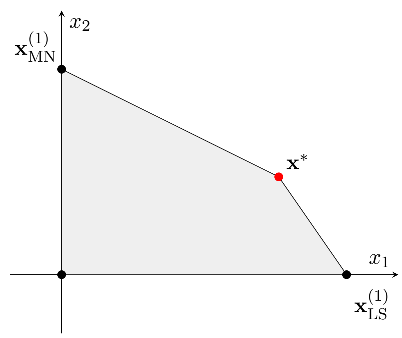
Week 3: Linear Programming, Feasible Region, Simplex Method

Consider the LP

$$\begin{aligned} \text{maximize} \quad & z = 4x_1 + 5x_2 \\ \text{subject to} \quad & 14x_1 + 11x_2 \le 154, \\ & 7x_1 + 16x_2 \le 112, \\ & x_i \ge 0, \ \forall i \in \{1, 2\} \end{aligned}$$

(a) **Feasible region.** The boundary lines are $14x_1 + 11x_2 = 154$ (intercepts $(11, 0)$ and $(0, 14)$) and $7x_1 + 16x_2 = 112$ (intercepts $(16, 0)$ and $(0, 7)$). In the first quadrant, the feasible polygon has vertices

$$(0,0), \ (11,0), \ \left(\frac{176}{21}, \frac{10}{3}\right), \ (0,7).$$



(b) **Dictionary.** Add slack variables x_3 and x_4:

$$\begin{aligned} \text{maximize} \quad & z = 4x_1 + 5x_2 \\ \text{subject to} \quad & x_3 = 154 - 14x_1 - 11x_2, \\ & x_4 = 112 - 7x_1 - 16x_2, \\ & x_i \ge 0, \ \forall i \in \{1, 2, 3, 4\} \end{aligned}$$

The corresponding initial tableau (columns ordered as $x_1, x_2, x_3, x_4, z \mid b$) is

$$T^{(0)} = \begin{array}{rrrrr|r} 14 & 11 & 1 & 0 & 0 & 154 \\ 7 & 16 & 0 & 1 & 0 & 112 \\ \hline -4 & -5 & 0 & 0 & 1 & 0 \end{array}$$

with basis $\beta^{(0)} = \{3, 4\}$ and nonbasic variables $\pi^{(0)} = \{1, 2\}$. The basic solution is

$$\mathbf{x}^{(0)} = (0, 0, 154, 112)^T, \qquad z^{(0)} = 0.$$

(c) **Simplex using the least-subscript rule (pivot columns).**

Iteration 0 → 1. In the objective row of $T^{(0)}$, both coefficients in columns 1 and 2 are negative, so increasing x_1 or x_2 increases z. The least-subscript rule picks x_1 as the entering variable (pivot column 1).

Ratio test (using the positive entries in column 1):

$$\frac{154}{14} = 11, \qquad \frac{112}{7} = 16.$$

The tightest restriction is row 1, so x_3 leaves. Pivot on the entry 14.

The resulting tableau is

$$T^{(1)} = \begin{array}{ccccc|c} 1 & \frac{11}{14} & \frac{1}{14} & 0 & 0 & 11 \\ 0 & \frac{21}{2} & -\frac{1}{2} & 1 & 0 & 35 \\ \hline 0 & -\frac{13}{7} & \frac{2}{7} & 0 & 1 & 44 \end{array}$$

so

$$\beta^{(1)} = \{1, 4\}, \quad \pi^{(1)} = \{2, 3\}, \quad \mathbf{x}^{(1)} = (11, 0, 0, 35)^T, \quad z^{(1)} = 44.$$

Iteration 1 → 2. In $T^{(1)}$, column 2 has a negative objective coefficient, so x_2 enters. Ratio test (positive entries in column 2):

$$\frac{11}{11/14} = 14, \qquad \frac{35}{21/2} = \frac{10}{3}.$$

The tightest restriction is row 2, so x_4 leaves. Pivot on the entry 21/2.

The resulting tableau is

$$T^{(2)} = \begin{array}{ccccc|c} 1 & 0 & \frac{16}{147} & -\frac{11}{147} & 0 & \frac{176}{21} \\ 0 & 1 & -\frac{1}{21} & \frac{2}{21} & 0 & \frac{10}{3} \\ \hline 0 & 0 & \frac{29}{147} & \frac{26}{147} & 1 & \frac{1054}{21} \end{array}$$

Now the objective row has no negative coefficients in the nonbasic columns, so the algorithm halts. Thus

$$\beta^{(2)} = \{1, 2\}, \quad \pi^{(2)} = \{3, 4\}, \quad \mathbf{x}^{(2)} = \left(\frac{176}{21}, \frac{10}{3}, 0, 0\right)^T, \quad z^{(2)} = \frac{1054}{21}.$$

(d) **Plot the iterates.** On the feasible-region plot from (a), mark the points

$$(x_1, x_2)^{(0)} = (0, 0), \qquad (x_1, x_2)^{(1)} = (11, 0), \qquad (x_1, x_2)^{(2)} = \left(\frac{176}{21}, \frac{10}{3}\right).$$

(These are the successive basic feasible solutions visited by the simplex method.)

(e) **Repeat (c)–(d) using the most-negative rule.**

At $T^{(0)}$, the most negative objective coefficient is in column 2 (since $-5 < -4$), so x_2 enters first. Ratio test in column 2:

$$\frac{154}{11} = 14, \qquad \frac{112}{16} = 7,$$

so row 2 is the tightest restriction and x_4 leaves (pivot on 16). This gives

$$\beta^{(1)}_{\text{MN}} = \{2, 3\}, \quad \mathbf{x}^{(1)}_{\text{MN}} = (0, 7, 77, 0)^T, \quad z^{(1)}_{\text{MN}} = 35.$$

In the next tableau, column 1 has the only negative objective coefficient, so x_1 enters and (by the ratio test) x_3 leaves, producing the same optimal tableau $T^{(2)}$ as above. Hence the most-negative method visits

$$(0, 0) \;\rightarrow\; (0, 7) \;\rightarrow\; \left(\frac{176}{21}, \frac{10}{3}\right)$$

with objective values

$$0 \;\rightarrow\; 35 \;\rightarrow\; \frac{1054}{21}.$$