

## Math 482 Workshop

### Week 4: Linear Programming, Simplex Method, Geometry of the Feasible Region

**Instructions.** Write clear solutions on your own paper. Show enough work to justify your answers.

I. Consider the LP shown below.

$$\begin{aligned} \text{maximize} \quad & z = x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 + 4x_2 - x_3 \leq 1, \\ & 4x_1 + x_2 - x_3 \leq 1, \\ & 2x_1 + 3x_2 + x_3 \leq 2, \\ & x_i \geq 0, \forall i \in \{1, 2, 3\} \end{aligned}$$

- Draw the feasible region.
- Next, add planes corresponding to  $x_i = 0$ , for  $1 \leq i \leq 6$ , to your plot. Note that, for  $i \geq 4$ ,  $x_i$  is a slack variable not explicitly represented on the graph. However, setting a slack variable equal to zero corresponds to the equation of a plane; for example,  $x_4 = 0$  corresponds to

$$x_1 + 4x_2 - x_3 = 1.$$

- Each extreme point of the feasible region is the intersection of three planes. For each extreme point, identify the corresponding three planes.
- Each extreme point of the feasible region corresponds to a feasible basic solution. For each extreme point, identify the corresponding basic solution.

II. Let  $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ , where

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Also, let  $\gamma = \sum_{j=1}^4 \frac{1}{4} \mathbf{x}_j$ .

- Draw the set  $X$  and  $\gamma$ .
- On the same graph, plot the line segment from  $\mathbf{x}_2$  to  $\mathbf{x}_4$ . Use this plot to determine  $\gamma$  as a convex combination of  $\mathbf{x}_2$  and  $\mathbf{x}_4$ .
- Use the proof technique from Carathéodory's Theorem to determine  $\gamma$  as a convex combination of at most three points from  $X$ .

III. Let  $P \subset \mathbb{R}^n$  be a polytope with extreme points  $X = \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ . We know that  $\text{convHull}(X) \subseteq P$ . Suppose that there is a  $\mathbf{b} \in P$  such that  $\mathbf{b} \notin \text{convHull}(X)$ . Use Farkas' Lemma (see Wikipedia page) to show that there exists a  $\mathbf{y} \in \mathbb{R}^n$  and  $\alpha \in \mathbb{R}$  such that  $\mathbf{b}^T \mathbf{y} + \alpha < 0$  and

$$\mathbf{x}_i^T \mathbf{y} + \alpha \geq 0, \quad 1 \leq i \leq k.$$

Now, consider the following maximization problem:

$$\begin{aligned} \text{maximize} \quad & z = -\mathbf{y}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{x} \in P \end{aligned}$$

Can you identify a contradiction?

IV. Let  $P \subset \mathbb{R}^n$  be a polytope with extreme points  $X = \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ . Consider the model

$$\begin{aligned} & \text{maximize} && z = \mathbf{c}^T \mathbf{x} \\ & \text{subject to} && \mathbf{x} \in P \end{aligned}$$

Prove that this model has an optimal value that is attained at an extreme point of  $P$ .