Math 482 Workshop — Solutions

Week 4: Linear Programming, Simplex Method, Geometry of the Feasible Region

Instructions. Write clear solutions on your own paper. Show enough work to justify your answers.

I. Consider the LP shown below.

$$\begin{aligned} \text{maximize} \quad & z = x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 + 4x_2 - x_3 \le 1, \\ & 4x_1 + x_2 - x_3 \le 1, \\ & 2x_1 + 3x_2 + x_3 \le 2, \\ & x_i \ge 0, \ \forall i \in \{1,2,3\} \end{aligned}$$

Introduce slack variables $x_4, x_5, x_6 \ge 0$:

$$\begin{aligned} x_1 + 4x_2 - x_3 + x_4 &= 1, \\ 4x_1 + x_2 - x_3 + x_5 &= 1, \\ 2x_1 + 3x_2 + x_3 + x_6 &= 2. \end{aligned}$$

a. **Solution.** The feasible region is the polytope in $\mathbb{R}^3$ cut out by the above three halfspaces together with $x_1, x_2, x_3 \ge 0$. (For a sketch: plot the three bounding planes and the coordinate planes in Desmos/GeoGebra.)

b. **Solution.** The six planes $x_i = 0$ for $1 \le i \le 6$ are

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = 0, \quad x_6 = 0.$$

For $i \ge 4$, $x_i = 0$ is a slack condition; e.g. $x_4 = 0$ means $x_1 + 4x_2 - x_3 = 1$.

c. **Solution.** Each extreme point is the intersection of three planes from the list above. Solving the feasible intersections gives:

(x_1, x_2, x_3)	three active planes
$(0,0,0)$	$x_1 = 0,\ x_2 = 0,\ x_3 = 0$
$(0,0,2)$	$x_1 = 0,\ x_2 = 0,\ x_6 = 0$
$(\frac{1}{4},0,0)$	$x_2 = 0,\ x_3 = 0,\ x_5 = 0$
$(0,\frac{1}{4},0)$	$x_1 = 0,\ x_3 = 0,\ x_4 = 0$
$(\frac{1}{5},\frac{1}{5},0)$	$x_3 = 0,\ x_4 = 0,\ x_5 = 0$
$(\frac{1}{2},0,1)$	$x_2 = 0,\ x_5 = 0,\ x_6 = 0$
$(0,\frac{3}{7},\frac{5}{7})$	$x_1 = 0,\ x_4 = 0,\ x_6 = 0$
$(\frac{3}{10},\frac{3}{10},\frac{1}{2})$	$x_4 = 0,\ x_5 = 0,\ x_6 = 0$

d. **Solution.** At any feasible point (x_1, x_2, x_3), the slack variables are

$$x_4 = 1 - (x_1 + 4x_2 - x_3), \qquad x_5 = 1 - (4x_1 + x_2 - x_3), \qquad x_6 = 2 - (2x_1 + 3x_2 + x_3).$$

Evaluating at each extreme point gives the corresponding feasible basic solution $(x_1, \ldots, x_6)$:

(x_1,x_2,x_3)	$(x_1,x_2,x_3,x_4,x_5,x_6)$
$(0,0,0)$	$(0,0,0,1,1,2)$
$(0,0,2)$	$(0,0,2,3,3,0)$
$(\frac{1}{4},0,0)$	$(\frac{1}{4},0,0,\frac{3}{4},0,\frac{3}{2})$
$(0,\frac{1}{4},0)$	$(0,\frac{1}{4},0,0,\frac{3}{4},\frac{5}{4})$
$(\frac{1}{5},\frac{1}{5},0)$	$(\frac{1}{5},\frac{1}{5},0,0,0,\frac{6}{5})$
$(\frac{1}{2},0,1)$	$(\frac{1}{2},0,1,\frac{3}{2},0,0)$
$(0,\frac{3}{7},\frac{5}{7})$	$(0,\frac{3}{7},\frac{5}{7},0,\frac{9}{7},0)$
$(\frac{3}{10},\frac{3}{10},\frac{1}{2})$	$(\frac{3}{10},\frac{3}{10},\frac{1}{2},0,0,\frac{1}{5})$

II. Let $X = \{x_1, x_2, x_3, x_4\}$, where

$$x_1 = \begin{bmatrix}0\\0\end{bmatrix}, \quad x_2 = \begin{bmatrix}1\\2\end{bmatrix}, \quad x_3 = \begin{bmatrix}3\\2\end{bmatrix}, \quad x_4 = \begin{bmatrix}2\\0\end{bmatrix}.$$

Let $\gamma = \frac{1}{4}(x_1 + x_2 + x_3 + x_4)$.

(a) **Solution.**

$$\gamma = \frac{1}{4}\begin{bmatrix}0+1+3+2\\0+2+2+0\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\1\end{bmatrix}.$$

(b) **Solution.** The line segment from x_2 to x_4 is $(1-\lambda)x_2 + \lambda x_4$ for $0 \le \lambda \le 1$. Solve

$$(1-\lambda)\begin{bmatrix}1\\2\end{bmatrix} + \lambda\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}1+\lambda\\2-2\lambda\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\1\end{bmatrix}.$$

From $1+\lambda = \frac{3}{2}$ we get $\lambda = \frac{1}{2}$, hence

$$\gamma = \tfrac{1}{2}x_2 + \tfrac{1}{2}x_4.$$

(c) **Solution.** (Carathéodory LP in $\mathbb{R}^2$.) Write $\gamma = t_1x_1 + t_2x_2 + t_3x_3 + t_4x_4$ with $t_j \ge 0$ and $t_1 + t_2 + t_3 + t_4 = 1$. As in the proof of Carathéodory's Theorem, the coefficients $t = (t_1, t_2, t_3, t_4)$ form a feasible solution of the LP

$$\begin{aligned}
&\text{find } t_1, t_2, t_3, t_4\\
&\text{s.t. } t_1x_{i1} + t_2x_{i2} + t_3x_{i3} + t_4x_{i4} = \gamma_i, \quad i = 1, 2,\\
&\phantom{\text{s.t. }} t_1 + t_2 + t_3 + t_4 = 1,\\
&\phantom{\text{s.t. }} t_j \ge 0, \quad j = 1, 2, 3, 4,
\end{aligned}$$

where x_{ij} denotes the ith coordinate of x_j.

Since we are in $\mathbb{R}^2$, there are $n+1 = 3$ equality constraints, so any feasible *basic* solution has at most 3 nonzero coefficients. Take $t_3 = 0$ as the nonbasic variable and solve:

$$\begin{aligned}
t_2 + 2t_4 &= \tfrac{3}{2},\\
2t_2 &= 1,\\
t_1 + t_2 + t_4 &= 1.
\end{aligned}$$

From $2t_2 = 1$ we have $t_2 = \frac{1}{2}$. Then $t_2 + 2t_4 = \frac{3}{2}$ gives $t_4 = \frac{1}{2}$, and the sum constraint gives $t_1 = 0$. Hence the feasible basic solution is

$$(t_1, t_2, t_3, t_4) = (0, \tfrac{1}{2}, 0, \tfrac{1}{2}),$$

so

$$\gamma = \tfrac{1}{2}x_2 + \tfrac{1}{2}x_4,$$

a convex combination of at most three points from X (in fact, two).

III. Let $P \subset \mathbb{R}^n$ be a polytope with extreme points $X = \{x_1, \ldots, x_k\}$. We know $\text{convHull}\,(X) \subseteq P$. Suppose there is $b \in P$ such that $b \notin \text{convHull}\,(X)$.

Solution. Since $\text{convHull}\,(X)$ is convex and $b \notin \text{convHull}\,(X)$, a separating hyperplane exists. Equivalently (via Farkas' Lemma / separation), there exist $y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that

$$x_i^T y + \alpha \geq 0 \quad (1 \leq i \leq k), \qquad \text{but} \qquad b^T y + \alpha < 0.$$

Consider the LP

$$\max\{-y^T x : \ x \in P\}.$$

Because P is a polytope, this maximum is attained at some $x^\star \in P$. By the extreme point theorem (or Problem IV), we may take $x^\star$ to be an extreme point of P, so $x^\star = x_i$ for some i. But $x_i^T y + \alpha \geq 0$ implies $-y^T x_i \leq \alpha$, while $b^T y + \alpha < 0$ implies $-y^T b > \alpha$. Thus

$$-y^T b > -y^T x_i \quad \text{for every } i = 1, \ldots, k,$$

so b gives a strictly larger objective value than every extreme point, contradicting that an optimizer can be chosen at an extreme point.

IV. Let $P \subset \mathbb{R}^n$ be a polytope with extreme points $X = \{x_1, \ldots, x_k\}$. Consider

$$\max\{c^T x : \ x \in P\}.$$

Solution. By Problem III (and the notes), we have

$$P = \text{convHull}\,(X)\,.$$

Let $x^\star \in P$ be an optimal solution and write it as a convex combination of extreme points:

$$x^\star = \sum_{i=1}^{k} \lambda_i x_i, \qquad \lambda_i \geq 0, \qquad \sum_{i=1}^{k} \lambda_i = 1.$$

Apply $c^T(\cdot)$:

$$c^T x^\star = \sum_{i=1}^{k} \lambda_i \, c^T x_i.$$

Let $z^\star = \max\{c^T x : \ x \in P\}$. Since each $x_i \in P$, we have $c^T x_i \leq z^\star$ for all i. But the right-hand side is a convex combination of the numbers $c^T x_i$ and equals $c^T x^\star = z^\star$. A convex combination can equal the maximum only if every term with nonzero weight attains the same maximum. Therefore,

$$\lambda_i > 0 \implies c^T x_i = z^\star.$$

In particular, at least one $\lambda_i > 0$, so there exists an extreme point x_i with $c^T x_i = z^\star$. Hence the maximum is attained at an extreme point, and moreover every extreme point appearing in the convex combination for $x^\star$ (those with $\lambda_i > 0$) is also optimal.