Math 482 Workshop — Solutions

Week 6: Duality Certificates and Complementary Slackness

Instructions. Write clear solutions on your own paper. Show enough work to justify your answers.

I. **(a)** The dual of P is

$$\begin{aligned} \text{minimize} \quad & w = 5y_1 + 22y_2 + 8y_3 \\ \text{subject to} \quad & y_1 - y_2 + 3y_3 \geq 2, \\ & 4y_2 + y_3 \geq 4, \\ & y_1 + 2y_2 + y_3 \geq 5, \\ & 4y_1 - 3y_2 + y_3 \geq 8, \\ & y_i \geq 0, \ \forall i \in \{1,2,3\} \end{aligned}$$

(b) Primal feasibility:

$$x_1^\star + x_3^\star + 4x_4^\star = 0 + 5 + 0 = 5, \qquad -x_1^\star + 4x_2^\star + 2x_3^\star - 3x_4^\star = 0 + 12 + 10 - 0 = 22,$$

$$3x_1^\star + x_2^\star + x_3^\star + x_4^\star = 0 + 3 + 5 + 0 = 8,$$

and $\mathbf{x}^\star \geq 0$. So $\mathbf{x}^\star$ is primal feasible.

Dual feasibility: compute

$$A^T\mathbf{y}^\star = \begin{bmatrix} y_1^\star - y_2^\star + 3y_3^\star \\ 4y_2^\star + y_3^\star \\ y_1^\star + 2y_2^\star + y_3^\star \\ 4y_1^\star - 3y_2^\star + y_3^\star \end{bmatrix} = \begin{bmatrix} 1 - 0 + 12 \\ 0 + 4 \\ 1 + 0 + 4 \\ 4 - 0 + 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 4 \\ 5 \\ 8 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 4 \\ 5 \\ 8 \end{bmatrix} = c.$$

Also $\mathbf{y}^\star \geq 0$. Hence $\mathbf{y}^\star$ is dual feasible.

(c) First compute primal slacks:

$$b - A\mathbf{x}^\star = \begin{bmatrix} 5 \\ 22 \\ 8 \end{bmatrix} - \begin{bmatrix} 5 \\ 22 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus $y_i^\star(b_i - (A\mathbf{x}^\star)_i) = 0$ holds for $i = 1, 2, 3$.

Next compute dual slacks:

$$A^T\mathbf{y}^\star - c = \begin{bmatrix} 13 - 2 \\ 4 - 4 \\ 5 - 5 \\ 8 - 8 \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$x_1^\star \cdot 11 = 0, \quad x_2^\star \cdot 0 = 0, \quad x_3^\star \cdot 0 = 0, \quad x_4^\star \cdot 0 = 0.$$

Hence complementary slackness holds, and $\mathbf{x}^\star$ and $\mathbf{y}^\star$ are primal–dual optimal.

(d) There are two places where "both sides" hold:

- For the dual variable $y_2^\star$, we have $y_2^\star = 0$ *and* the corresponding primal constraint is tight (its slack is 0).
- For the primal variable $x_4^\star$, we have $x_4^\star = 0$ *and* the corresponding dual constraint is tight (its slack is 0).

II. **(a)** The dual of P is

$$\begin{aligned} \text{minimize} \quad & w = 455y_1 + 190y_2 + 205y_3 + 80y_4 - 200y_5 \\ \text{subject to} \quad & 8y_1 + 5y_2 + 4y_3 - y_4 - 3y_5 \geq 13, \\ & 7y_1 - y_2 + 8y_3 + 2y_4 - 5y_5 \geq 20, \\ & 9y_1 + 6y_2 - y_3 + 3y_4 - 4y_5 \geq 17, \\ & \qquad\qquad y_i \geq 0, \ \forall i \in \{1,2,3,4,5\} \end{aligned}$$

(b) We test feasibility and then (when possible) complementary slackness / equality of objective values.

(i) For $\mathbf{x}' = \frac{1}{79}(0, 2300, 2205)^T$, the fourth constraint evaluates to

$$-x_1' + 2x_2' + 3x_3' = \frac{1}{79}(0 + 4600 + 6615) = \frac{11215}{79} > 80,$$

so $\mathbf{x}'$ is not primal feasible. Therefore this pair cannot be a primal–dual optimal pair.

(ii) For $\mathbf{x}' = \frac{1}{180}(4600, 2600, 2300)^T$, direct substitution shows that all five constraints are satisfied and the second, third, and fifth constraints are tight:

$$5x_1' - x_2' + 6x_3' = 190, \qquad 4x_1' + 8x_2' - x_3' = 205, \qquad -3x_1' - 5x_2' - 4x_3' = -200.$$

Also $\mathbf{y}' = \frac{1}{180}(0, 753, 546, 0, 0)^T$ satisfies the dual constraints, so both vectors are feasible. However, the objective values are

$$c^T\mathbf{x}' = \frac{2515}{3} \qquad \text{and} \qquad b^T\mathbf{y}' = \frac{4250}{3}.$$

Since $c^T\mathbf{x}' \neq b^T\mathbf{y}'$, weak duality implies this pair is not optimal (and complementary slackness must fail).

(iii) For $\mathbf{x}' = \frac{1}{275}(4895, 5280, 5445)^T$, one checks that all five primal constraints are satisfied, and constraints 1, 3, and 4 are tight:

$$8x_1' + 7x_2' + 9x_3' = 455, \qquad 4x_1' + 8x_2' - x_3' = 205, \qquad -x_1' + 2x_2' + 3x_3' = 80.$$

For $\mathbf{y}' = \frac{1}{275}(390, 0, 230, 465, 0)^T$, one checks that each dual constraint is satisfied with equality:

$$A^T\mathbf{y}' = (13, 20, 17)^T.$$

Finally, the objective values match:

$$c^T\mathbf{x}' = 952 \qquad \text{and} \qquad b^T\mathbf{y}' = 952.$$

Therefore $\mathbf{x}'$ is primal optimal and $\mathbf{y}'$ is dual optimal.