# Math 482 Workshop

## Week 8: Matrix Form and Network Problems (Solutions)

**Instructions.** These solutions are written in a concise "handout" style. Minor variations in pivot choices are possible.

I. Let $G = (V, E)$ be a graph and consider the matching problem

$$\text{maximize} \quad z = \sum_{e \in E} x_e \tag{1a}$$

$$\text{subject to} \quad \sum_{e \in E:\, v \in e} x_e \leq 1, \ \forall v \in V, \tag{1b}$$

$$x_e \in \{0, 1\}, \ \forall e \in E. \tag{1c}$$

a. **Fractional matching interpretation.** The relaxation replaces $x_e \in \{0, 1\}$ with $0 \leq x_e \leq 1$ for each $e \in E$. Thus, instead of selecting a set of pairwise-disjoint edges, the LP can *split* across edges: each vertex $v$ has "capacity" 1 that can be shared fractionally among incident edges. An optimal fractional solution assigns weights to edges so that no vertex receives total weight more than 1, while maximizing the total assigned edge-weight $\sum_{e \in E} x_e$. Equivalently, it packs as many edges as possible, but is allowed to take an "edge at level 1/2", etc., as long as no vertex is used more than 1 in total.

b. **Dual of the fractional matching LP.** Let the primal be

$$\max \sum_{e \in E} x_e \quad \text{s.t.} \quad \sum_{e \ni v} x_e \leq 1 \ \ (\forall v \in V), \qquad x_e \geq 0 \ \ (\forall e \in E).$$

Introduce a dual variable $y_v \geq 0$ for each vertex constraint $v \in V$. Then the dual is

$$\min \sum_{v \in V} y_v \quad \text{s.t.} \quad y_u + y_v \geq 1 \ \ (\forall e = \{u, v\} \in E), \qquad y_v \geq 0 \ \ (\forall v \in V).$$

(Each edge-variable $x_{\{u,v\}}$ appears in exactly two vertex constraints, hence the dual constraint $y_u + y_v \geq 1$.)

c. **Integral dual variables.** If we additionally restrict $y_v \in \{0, 1\}$ (or more generally $y_v \in \mathbb{Z}_{\geq 0}$, which will automatically give 0/1 at optimum), then the dual is modeling a *minimum vertex cover*: choose a minimum-cardinality set $C \subseteq V$ so that every edge has at least one endpoint in $C$. Indeed, $y_v = 1$ corresponds to selecting $v$ into the cover, and $y_u + y_v \geq 1$ forces each edge to be covered.

II. Consider the graph below (with bipartition $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3\}$ and edges $u_1v_1, u_1v_2, u_2v_1, u_2v_3, u_3v_2, u_3v_3$).

a. **Matching model.** Let the decision variables correspond to edges:

$$x_{11}, x_{12}, x_{21}, x_{23}, x_{32}, x_{33} \ \in \{0, 1\},$$

where $x_{ij} = 1$ means we select edge $u_i v_j$. Then the matching problem is

$$\begin{aligned}
\text{maximize} \quad & z = x_{11} + x_{12} + x_{21} + x_{23} + x_{32} + x_{33} && \text{(2a)}\\
\text{subject to} \quad & x_{11} + x_{12} \le 1 \quad \text{(vertex } u_1\text{)}, && \text{(2b)}\\
& x_{21} + x_{23} \le 1 \quad \text{(vertex } u_2\text{)}, && \text{(2c)}\\
& x_{32} + x_{33} \le 1 \quad \text{(vertex } u_3\text{)}, && \text{(2d)}\\
& x_{11} + x_{21} \le 1 \quad \text{(vertex } v_1\text{)}, && \text{(2e)}\\
& x_{12} + x_{32} \le 1 \quad \text{(vertex } v_2\text{)}, && \text{(2f)}\\
& x_{23} + x_{33} \le 1 \quad \text{(vertex } v_3\text{)}, && \text{(2g)}\\
& x_{ij} \in \{0,1\} \text{ for each listed edge.} && \text{(2h)}
\end{aligned}$$

The *fractional matching* relaxation replaces the last line by $x_{ij} \ge 0$ (and optionally $x_{ij} \le 1$).

b. **Total unimodularity and integrality.** Let $A$ be the $6 \times 6$ node–edge incidence matrix for the above constraints in the order

$$(u_1, u_2, u_3, v_1, v_2, v_3) \quad \text{and} \quad (x_{11}, x_{12}, x_{21}, x_{23}, x_{32}, x_{33}).$$

Then each column has exactly two 1's: one in the corresponding $u_i$-row and one in the corresponding $v_j$-row.

To see $A$ is totally unimodular, multiply the last three rows (the $V$-rows) by $-1$ to obtain a matrix $\widetilde{A}$ whose columns each have exactly one $+1$ and exactly one $-1$. Thus, $\widetilde{A}$ is the node–arc incidence matrix of a directed graph (orient each edge from $U$ to $V$). Incidence matrices of directed graphs are totally unimodular, hence $\widetilde{A}$ is TU, and therefore $A$ is TU as well (row-sign changes preserve total unimodularity).

**Why this implies integrality.** In the fractional matching LP we have $Ax \le \mathbf{1}$ with $x \ge 0$ and integer right-hand side. If $A$ is totally unimodular, then every basic feasible solution of this LP is integral (in fact, 0/1 here), so an optimal solution can be chosen integral. Therefore the fractional matching optimum equals the (integral) maximum matching size.

c. **Simplex method in matrix form (one valid pivot sequence).** Write the fractional matching LP in standard form by introducing slack variables

$$s_{u_1}, s_{u_2}, s_{u_3}, s_{v_1}, s_{v_2}, s_{v_3} \ge 0$$

so that

$$Ax + s = \mathbf{1}, \qquad x \ge 0,\ s \ge 0, \qquad \max\ z = \mathbf{1}^T x.$$

We use the variable order

$$(x_{11}, x_{12}, x_{21}, x_{23}, x_{32}, x_{33}, s_{u_1}, s_{u_2}, s_{u_3}, s_{v_1}, s_{v_2}, s_{v_3}).$$

Below, the reduced cost row is reported using the convention

$$\bar{c}_\pi^T = c_\beta^T B^{-1} \Pi - c_\pi^T,$$

so *negative* entries indicate improving entering variables (for a maximization problem).

**Iteration** 0. Take the initial basis $\beta(0) = \{s_{u_1}, s_{u_2}, s_{u_3}, s_{v_1}, s_{v_2}, s_{v_3}\}$ and parameters $\pi(0) = \{x_{11}, x_{12}, x_{21}, x_{23}, x_{32}, x_{33}\}$. Then

$$x_\beta = \mathbf{1}, \quad x_\pi = \mathbf{0}, \quad z = 0,$$

and

$$\bar{c}_\pi^T = (-1, -1, -1, -1, -1, -1).$$

A convenient entering variable is $x_{12}$.

**Iteration 1.** Pivot $x_{12}$ into the basis and let $s_{u_1}$ leave. One obtains

$$x_{12} = 1, \qquad s_{u_2} = 1,\ s_{u_3} = 1,\ s_{v_1} = 1,\ s_{v_3} = 1, \qquad s_{v_2} = 0,$$

so $z = 1$. The associated dual vector (from $y^T = c_\beta^T B^{-1}$) is

$$y = (1, 0, 0, 0, 0, 0)^T$$

(corresponding to constraints $(u_1, u_2, u_3, v_1, v_2, v_3)$). The primal slack vector is

$$b - Ax = (0, 1, 1, 1, 0, 1)^T.$$

The dual slack vector is

$$A^T y - c = (0, 0, -1, -1, -1, -1)^T,$$

and the reduced costs for the original $x$-variables are

$$\bar{c} = (0, 0, -1, -1, -1, -1).$$

Choose $x_{21}$ to enter next.

**Iteration 2.** Pivot $x_{21}$ into the basis and let $s_{u_2}$ leave. Then

$$x_{12} = 1,\ x_{21} = 1, \qquad s_{u_3} = 1,\ s_{v_3} = 1, \qquad s_{u_1} = s_{u_2} = s_{v_1} = s_{v_2} = 0,$$

so $z = 2$. Now

$$y = (1, 1, 0, 0, 0, 0)^T, \qquad b - Ax = (0, 0, 1, 0, 0, 1)^T,$$
$$A^T y - c = (0, 0, 0, 0, -1, -1)^T, \qquad \bar{c} = (0, 0, 0, 0, -1, -1).$$

Choose $x_{33}$ to enter.

**Iteration 3.** Pivot $x_{33}$ into the basis and let $s_{u_3}$ leave. Then

$$x_{12} = 1,\ x_{21} = 1,\ x_{33} = 1, \qquad s_{u_1} = s_{u_2} = s_{u_3} = s_{v_1} = s_{v_2} = s_{v_3} = 0,$$

so $z = 3$. Moreover,

$$y = (1, 1, 1, 0, 0, 0)^T, \qquad \bar{c} = (0, 0, 0, 0, 0, 0),$$

hence there are no improving directions and the simplex method halts with an optimal solution.

d. **Complementary slackness verification.** Consider the primal solution

$$x_{12}^* = x_{21}^* = x_{33}^* = 1, \qquad \text{all other } x_e^* = 0,$$

which has value $z^* = 3$.
From the final iteration above, the primal slack vector satisfies $s = b - Ax^* = \mathbf{0}$, so

$$y_i\, s_i = 0 \quad \text{for every dual variable } y_i \geq 0.$$

Also, the final dual slack vector satisfies $t = A^T y - c = \mathbf{0}$, so

$$x_e^*\, t_e = 0 \quad \text{for every primal variable } x_e^* \geq 0.$$

Thus, the complementary slackness conditions hold, which certifies optimality of $x^*$ (and of the associated dual $y$).

III. Consider the linear ordering problem described in (4a)–(4d) of the Network Model notes. The decision variables $x_{ij} \in \{0, 1\}$ form a subdigraph.

(a) **Why (4b) gives a spanning tournament.** Constraint (4b) enforces that for each unordered pair $\{i, j\}$ (with $i \neq j$), exactly one of the arcs $(i, j)$ or $(j, i)$ is selected in the subdigraph:

$$x_{ij} + x_{ji} = 1.$$

Therefore the subdigraph has *all* vertices (it is spanning) and has *exactly one directed edge* between every pair of distinct vertices (it is a tournament).

(b) **Why (4b)–(4c) gives an acyclic digraph.** The variables satisfying (4b) describe a tournament. In a tournament, *every* directed cycle contains a directed 3-cycle as a subcycle (one way to see this: take any directed cycle of minimum length; if its length is $\geq 4$, a chord between two nonconsecutive vertices creates a shorter directed cycle, contradicting minimality). Constraint (4c) forbids directed 3-cycles by imposing, for all distinct $i, j, k$,

$$x_{ij} + x_{jk} + x_{ki} \leq 2.$$

Hence no directed 3-cycle can occur, and therefore no directed cycle can occur at all. Thus the subdigraph is acyclic.