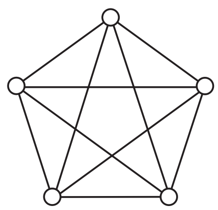
You should think of ν and ε as functions that, given a graph, return the number of vertices and edges, respectively.

The Greek letter ν (nu) corresponds to the Roman letter n (the usual letter for the number of vertices in a graph), and it looks like a v (for vertices). The Greek letter ε (a stylized epsilon) corresponds to the Roman e (for edges).



- *Complete graphs.*

  Let G be a graph. If all pairs of distinct vertices are adjacent in G, we call G *complete*. A complete graph on n vertices is denoted K_n. The graph in the figure is a K_5.

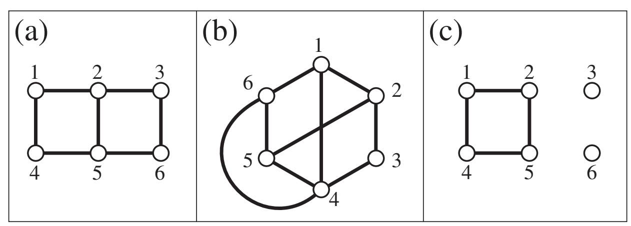
  The opposite extreme is a graph with no edges. We call such graphs *edgeless*.

Recap

We began by motivating the study of graph theory with three classic problems (and non-frivolous variations thereof). We then formally introduced the concept of a graph, being careful to distinguish between a graph and its drawing. We studied the adjacency relation, concluding with the result that the sum of the degrees of the vertices in a graph equals twice the number of edges in the graph. Finally, we introduced additional graph theory terminology.

47 Exercises

47.1. The following pictures represent graphs. Please write each of these graphs as a pair of sets (V, E).



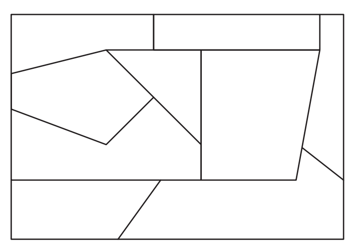
47.2. Draw pictures of the following graphs.

a. $\big(\{a,b,c,d,e\}, \big\{\{a,b\},\{a,c\},\{a,d\},\{b,e\},\{c,d\}\big\}\big)$.

b. $\big(\{a,b,c,d,e\}, \big\{\{a,b\},\{a,c\},\{b,c\},\{b,d\},\{c,d\}\big\}\big)$.

c. $\big(\{a,b,c,d,e\}, \big\{\{a,c\},\{b,d\},\{b,e\}\big\}\big)$.

47.3. Color the map in the figure with four colors (so that adjacent countries have different colors) and explain why it is not possible to color this map with only three colors.



47.4. In the map-coloring problem, why do we require that countries be connected (and not in multiple pieces like Russia or Michigan)?

Draw a map, in which disconnected countries are permitted, that requires more than four colors.

47.5. In the map-coloring problem, why do we allow countries that meet at only one point to receive the same color?

Draw a map that requires more than four colors if countries that meet only at one point must get different colors.

47.6. If three countries on a map all border each other, then the map certainly requires at least three colors. (For example, look at Brazil, Venezuela, and Colombia or at France, Germany, and Belgium.)

Devise a map in which no three countries border each other, and yet the map cannot be colored with fewer than three colors.

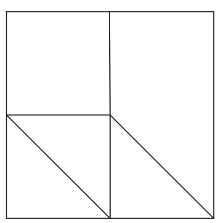
47.7. Imagine creating a map on your computer screen. This map wraps around the screen in the following way. A line that moves off the right side of the screen instantly reappears at the corresponding position on the left. Similarly, a line that drops off the bottom of the screen instantly reappears at the corresponding position at the top. Thus it is possible to have a country on this map that has a little section on the left and another little section on the right of the screen, but is still in one piece.

Devise such a computer-screen map that requires more than four colors.

Try to create such a map that requires seven colors! (It is possible.)

47.8. *Refer to the previous problem about drawing on your computer screen.* On this screen, can you solve the gas/water/electricity problem? That is, find a way to place the three utilities and the three houses such that the connecting lines don't cross. You may, of course, take advantage of the fact that a pipe can wrap from the left side of the screen across to the exact same point on the right or from top to bottom.

47.9. *Continued from the previous problem.* Suppose now you wish to add a cable television facility to your computer screen city. Can you run three television cables from the cable TV headquarters to each of the three houses without crossing any of the gas/water/electric lines?



47.10. Show how to draw the picture in the figure without lifting your pencil from the page and without redrawing any lines.

47.11. If you begin your drawing of the figure in Exercise 47.10 in the middle of the top, it's easy to obtain a solution. Show that is is possible to begin the drawing at the top middle point and yet, by making some unfortunate decisions, be unable to complete the drawing.

47.12. Recall the university examination-scheduling problem. Create a list of courses and students such that more than four final examination periods are required.

47.13. Construct a graph G for which the is-adjacent-to relation, $\sim$, is antisymmetric.

Construct a graph G for which the is-adjacent-to relation, $\sim$, is transitive.

47.14. In Definition 47.4 (degree), we defined $d(v)$ to be the number of edges incident with v. However, we also said that $d(v) = |N(v)|$. Why is this so?

Is $d(v) = |N(v)|$ true for a multigraph?

47.15. Let G be a graph. Prove that there must be an even number of vertices of odd degree. (For example, the graph in Example 47.2 has exactly four vertices of odd degree.)

47.16. Prove that in any graph with two or more vertices, there must be two vertices of the same degree.

47.17. Let G be an r-regular graph with n vertices and m edges. Find (and prove) a simple algebraic relation between r, n, and m.

47.18. Find all 3-regular graphs on nine vertices.

47.19. How many edges are in K_n, a complete graph on n vertices?

47.20. How many different graphs can be formed with vertex set $V = \{1, 2, 3, \ldots, n\}$?

47.21. What does it mean for two graphs to be the same? Let G and H be graphs. We say that G is *isomorphic* to H provided there is a bijection $f : V(G) \to V(H)$ such that for all $a, b \in V(G)$ we have $a \sim b$ (in G) if and only if $f(a) \sim f(b)$ (in H). The function f is called an *isomorphism* of G to H.

We can think of f as renaming the vertices of G with the names of the vertices in H in a way that preserves adjacency. Less formally, isomorphic graphs have the same drawing (except for the names of the vertices).

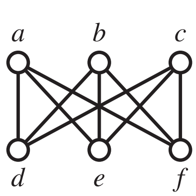
Please do the following:

a. Prove that isomorphic graphs have the same number of vertices.

b. Prove that if $f : V(G) \to V(H)$ is an isomorphism of graphs G and H and if $v \in V(G)$, then the degree of v in G equals the degree of $f(v)$ in H.

c. Prove that isomorphic graphs have the same number of edges.

d. Give an example of two graphs that have the same number of vertices and the same number of edges but that are not isomorphic.



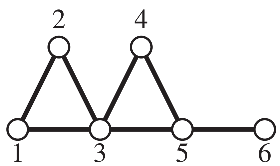
e. Let G be the graph whose vertex set is $\{1, 2, 3, 4, 5, 6\}$. In this graph, there is an edge from v to w if and only if $v - w$ is odd. Let H be the graph in the figure.

Find an isomorphism $f : V(G) \to V(H)$.

Recap

We introduced the concept of subgraph and the special forms of subgraph: spanning and induced. We discussed cliques and independent sets. We presented the concept of the complement of a graph. Finally, we presented a simplified version of Ramsey's Theorem.

48 Exercises



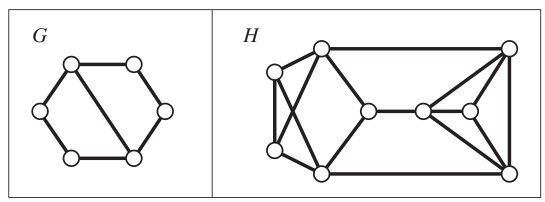
48.1. Let G be the graph in the figure. Draw pictures of the following subgraphs.
- **a.** $G - 1$.
- **b.** $G - 3$.
- **c.** $G - 6$.
- **d.** $G - \{1, 2\}$.
- **e.** $G - \{3, 5\}$.
- **f.** $G - \{5, 6\}$.
- **g.** $G[\{1, 2, 3, 4\}]$.
- **h.** $G[\{2, 4, 6\}]$.
- **i.** $G[\{1, 2, 4, 5\}]$.

48.2. Which of the various properties of relations does the is-a-subgraph-of relation exhibit? Is it reflexive? Irreflexive? Symmetric? Antisymmetric? Transitive?

48.3. Let C be a clique and let I be an independent set in a graph G. Prove that $|C \cap I| \le 1$.

48.4. Let G be a complete graph on n vertices.
- **a.** How many spanning subgraphs does G have?
- **b.** How many induced subgraphs does G have?

48.5. Let G and H be the two graphs in the figure.



Please find $\alpha(G)$, $\omega(G)$, $\alpha(H)$, and $\omega(H)$.

48.6. Find a graph G with $\alpha(G) = \omega(G) = 5$.

48.7. Suppose that G is a subgraph of H. Prove or disprove:
- **a.** $\alpha(G) \le \alpha(H)$.
- **b.** $\alpha(G) \ge \alpha(H)$.
- **c.** $\omega(G) \le \omega(H)$.
- **d.** $\omega(G) \ge \omega(H)$.

The graph in this exercise is an example of a *complete bipartite graph*. This particular complete bipartite graph is denoted $K_{3,5}$. This concept is formally introduced in Definition 52.10.

48.8. Let G be with $V(G) = X \cup Y$ where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$. Every vertex in X is adjacent to every vertex in Y, but there are no other edges in G.

Please do:
- **a.** Find all the maximal independent sets of G.
- **b.** Find all the maximum independent sets of G.
- **c.** Find all the maximal cliques of G.
- **d.** Find all the maximum cliques of G.

This problem involves a special case of Turán's Theorem which answers the following question: Given positive integers n and r, what is the maximum number of edges in a graph G with n vertices and $\omega(G) \le r$? In this problem, we seek the answer in the case $n = 100$ and $r = 2$.

48.9. Let G be a graph with $n = 100$ vertices that does not contain K_3 as a subgraph; in other words, $\omega(G) \le 2$. (Such graphs are called *triangle free*.) What can we say about the maximum number of edges in such a graph?

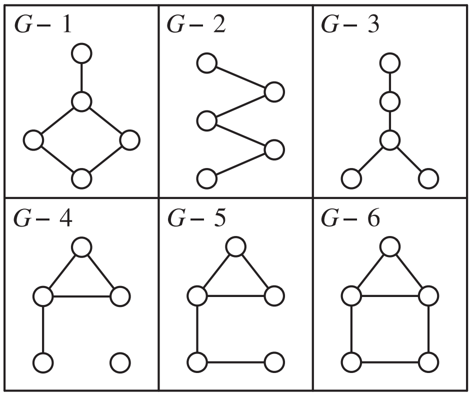
Imagine this problem as a contest. Your job is to build a triangle-free graph with as many edges as possible. To get the competition started, Alex says: "If I take one vertex and join it by edges to all the others, I can make a triangle-free graph with 99 edges. And it's not possible to add an edge to my graph without making a triangle!" But then Beth counters, "Yes, but if we just put all the vertices in a big cycle, we can make a triangle-free graph with 100 edges." Eve, who has been eavesdropping on their conversation, adds, "But then I can draw a diagonal edge across Betty's cycle and get 101 edges." Zeke, who has not been paying particularly close attention, wakes up and says, "My graph has 4950 edges, so I win!" Of course, he doesn't tell anyone how he got his answer.

You can do better than Eve's 101 edges—a lot better. Create a triangle-free graph with 100 vertices and as many edges as you can. If you like, try to prove that your graph is best possible.

In any case, prove that Zeke is wrong. (What was he thinking!?)

This is a special case of the *reconstruction problem*. In the general case, suppose there is an unknown graph G with n vertices where $n > 2$. We are given n unlabeled drawings of the graphs $G - v$; one for each $v \in V(G)$. The question is: Do these n drawings uniquely determine the graph G?

48.10. Let $G = (V, E)$ be a graph with $V = \{1, 2, 3, 4, 5, 6\}$. In the figure we show the graphs $G - 1$, $G - 2$, and so on but we do not show the names of the vertices.



The goal of this problem is to reconstruct the original graph G. Please do:

a. Determine the number of edges in G.

b. Using your answer to (a), determine the degrees of each of the six vertices of G.

c. Determine G.

Self-complementary graphs

48.11. Recall the definition of graph isomorphism from Exercise 47.21. We call a graph G *self-complementary* if G is isomorphic to $\overline{G}$.

a. Show that the graph $G = \big(\{a, b, c, d\}, \{ab, bc, cd\}\big)$ is self-complementary.

b. Find a self-complementary graph with five vertices.

c. Prove that if a self-complementary graph has n vertices, then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

48.12. Find a graph G on five vertices for which $\omega(G) < 3$ and $\omega(\overline{G}) < 3$. This shows that the number six in Proposition 48.13 is best possible.

48.13. Let G be a graph with at least two vertices. Prove that $\alpha(G) \geq 2$ or $\omega(G) \geq 2$.

Ramsey arrow notation

48.14. Let $n, a, b \geq 2$ be integers. The notation $n \to (a, b)$ is an abbreviation for the following sentence:

Every graph G on n vertices has $\alpha(G) \geq a$ or $\omega(G) \geq b$.

For example, Proposition 48.13 says that if $n \geq 6$, then $n \to (3, 3)$ is true. However, Exercise 48.12 asserts that $5 \to (3, 3)$ is false.

Please prove the following:

a. If $n \geq 2$, then $n \to (2, 2)$.

b. For any integer $n \geq 2$, $n \to (n, 2)$.

c. If $n \to (a, b)$ and $m \geq n$, then $m \to (a, b)$.

d. If $n \to (a, b)$, then $n \to (b, a)$.

e. The least n such that $n \to (3, 3)$ is $n = 6$.

f. $10 \to (3, 4)$.

g. Suppose $a, b \geq 3$. If $n \to (a - 1, b)$ and $m \to (a, b - 1)$, then $(n + m) \to (a, b)$.

h. $20 \to (4, 4)$.

49 Connection

Graphs are useful in modeling communication and transportation networks. The vertices of a graph can represent major cities in a country, and the edges in the graph can represent highways that link them. A fundamental question is: For a given pair of sites in the network, can we travel from one to the other?

For example, in the United States, we can travel by interstate from Baltimore to Denver, but we cannot get to Honolulu from Chicago, even though both of these cities are serviced by interstates. (Some so-called "interstate" highways actually reside entirely within one state, such as I-97 in Maryland and H-1 in Hawaii.)