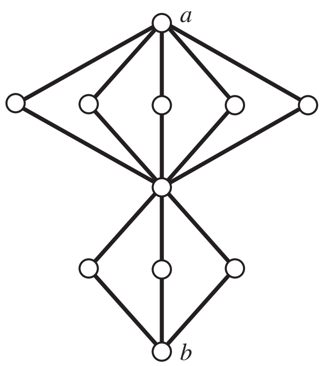
not use the edge e. Therefore there is a (c, a)-walk in $G - e$ and hence (Lemma 49.7) a (c, a)-walk in $G - e$. This contradicts the fact that in $G - e$ we have c and b in separate components.

Therefore $G - e$ has at most two components. ■

Recap

We began with the concepts of walk and path. From there, we defined what it means for a graph to be connected and what its connected components are. We discussed cut vertices and cut edges.

49 Exercises



49.1. Let G be the graph in the figure.
a. How many different paths are there from a to b?
b. How many different walks are there from a to b?

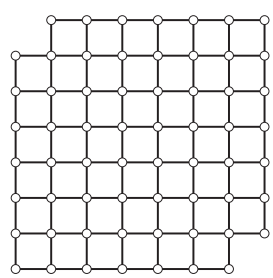
49.2. Is concatenation a commutative operation?

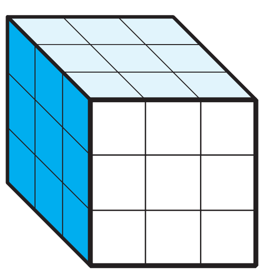
49.3. Prove that K_n is connected.

49.4. Let $n \geq 2$ be an integer. Form a graph G_n whose vertices are all the two-element subsets of $\{1, 2, \ldots, n\}$. In this graph we have an edge between distinct vertices $\{a, b\}$ and $\{c, d\}$ exactly when $\{a, b\} \cap \{c, d\} = \emptyset$.
Please answer:
a. How many vertices does G_n have?
b. How many edges does G_n have?
c. For which values of $n \geq 2$ is G_n connected? Prove your answer.

49.5. Consider the following (incorrect) restatement of the definition of *connected*: "A graph G is connected provided there is a path that contains every pair of vertices in G."
What is wrong with this sentence?

49.6. Let G be a graph. A path P in G that contains all the vertices of G is called a *Hamiltonian path*. Prove that the following graph does not have a Hamiltonian path.





49.7. How many Hamiltonian paths (see previous problem for definition) does a complete graph on $n \geq 2$ vertices have?

49.8. *Mouse and cheese*. A block of cheese is made up of $3 \times 3 \times 3$ cubes as in the figure. Is it possible for a mouse to tunnel its way through this block of cheese by (a) starting at a corner, (b) eating its way from cube to adjacent cube, (c) never passing though any cube twice, and, finally, (d) finishing at the center cube? Prove your answer.

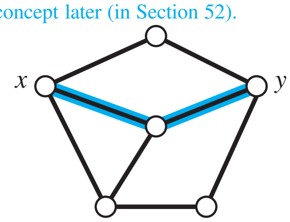
49.9. Consider the is-connected-to relation on the vertices of a graph. Show that is-connected-to need not be irreflexive or antisymmetric.

49.10. Let G be a graph. Prove that G or $\overline{G}$ (or both) must be connected.

49.11. Let G be a graph with $n \geq 2$ vertices. Prove that if $\delta(G) \geq \frac{1}{2}n$, then G is connected.

This exercise develops the notion of *distance* in graphs. We need this concept later (in Section 52).

49.12. Let G be a graph with $n \geq 2$ vertices.
a. Prove that if G has at least $\binom{n-1}{2} + 1$ edges, then G is connected.
b. Show that the result in (a) is best possible; that is, for each $n \geq 2$, prove there is a graph with $\binom{n-1}{2}$ edges that is not connected.



49.13. Let G be a graph and let $v, w \in V(G)$. The *distance* from v to w is the length of a shortest (v, w)-path and is denoted $d(v, w)$. In case there is no v, w-path, we may either say that $d(v, w)$ is undefined or infinite. For the graph in the figure, there are several (x, y)-paths; the shortest among them have length 2. Thus $d(x, y) = 2$.

Prove that graph distance satisfies the *triangle inequality*. That is, if x, y, z are vertices of a connected graph G, then

$$d(x,z) \le d(x,y) + d(y,z).$$

49.14. *For those who have studied linear algebra*. Let A be the adjacency matrix of a graph G. That is, we label the vertices of G as $v_1, v_2, \ldots, v_n$. The matrix A is an $n \times n$ matrix whose i,j-entry is 1 if $v_i v_j \in E(G)$ and is 0 otherwise.

Let $k \in \mathbb{N}$. Prove that the i,j-entry of A^k is the number of walks of length k from v_i to v_j.

49.15. Let n and k be integers with $1 \le k < n$. Form a graph G whose vertices are the integers $\{0, 1, 2, \ldots, n-1\}$. We have an edge joining vertices a and b provided

$$a - b \equiv \pm k \pmod{n}.$$

For example, if $n = 20$ and $k = 6$, then vertex 2 would be adjacent to vertices 8 and 16.

a. Find necessary and sufficient conditions on n and k such that G is connected.

b. Find a formula involving n and k for the number of connected components of G.

50 Trees

One of the simplest family of graphs are the *trees*. Graph theory problems can be difficult. Often, a good way to begin thinking about these problems is to solve them for trees. Trees are also the most basic connected graph. What are trees? They are connected graphs that have no cycles. We begin by defining the term *cycle*.

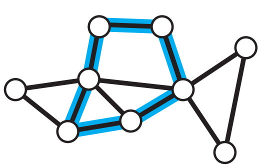
Cycles

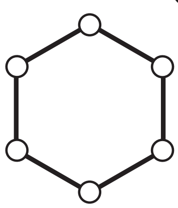
Definition 50.1 **(Cycle)** A *cycle* is a walk of length at least three in which the first and last vertex are the same, but no other vertices are repeated.

The term *cycle* also refers to a (sub)graph consisting of the vertices and edges of such a walk. In other words, a cycle is a graph of the form $G = (V, E)$ where

$$V = \{v_1, v_2, \ldots, v_n\} \quad \text{and}$$
$$E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}.$$

A cycle (graph) on n vertices is denoted C_n.





In the upper figure we see a cycle of length six as a subgraph of a graph. The lower figure shows the graph C_6.

Forests and Trees

Definition 50.2 **(Forest)** Let G be a graph. If G contains no cycles, then we call G *acyclic*. Alternatively, we call G a *forest*.

The term *acyclic* is more natural and (almost) does not need a definition—its standard English meaning is a perfect match for its mathematical usage. The term *forest* is widely used as well. The rationale for this word is that here, just as in real life, a forest is a collection of trees.

Definition 50.3 **(Tree)** A *tree* is a connected, acyclic graph.

In other words, a *tree* is a connected forest.

The forest in the figure contains four connected components. Each component of a forest is a tree.

Note that a single isolated vertex (e.g., the graph K_1) is a tree.