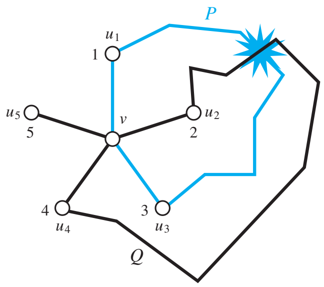


The problem, as before, is that perhaps u_2 and u_4 are in the same component of $H_{2,4}$. We claim, however, that this cannot happen! Suppose there is a path, Q, from u_2 to u_4. Note that the vertices along Q are colored with colors 2 and 4, and the vertices on P are colored with colors 1 and 3. Thus P and Q have no vertices in common. Furthermore, path P, together with vertex v, forms a cycle. This cycle becomes a simple closed curve in the plane. Notice that vertices u_2 and u_4 are on different sides of this curve! Therefore the path Q from u_2 to u_4 must pass from the inside of this simple closed curve to the outside, and where it does, there is an edge crossing. However, by construction, this embedding has no edge crossings! Therefore vertices u_2 and u_4 must be in separate components of $H_{2,4}$, and the 2-for-4 recoloring technique may be used. Finally, we color vertex v with color 2, giving a proper five-coloring of G. ■

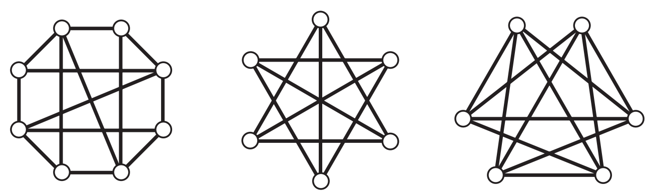
Recap

We introduced the concept of planar graphs: graphs that can be drawn in the plane without edges crossing. We presented Euler's formula that relates the number of vertices, edges, and faces of a connected planar graph and used it to find bounds on the number of edges in a planar graph. We showed that K_5 and $K_{3,3}$ are nonplanar and discussed Kuratowski's Theorem, which says, in essence, that these two graphs are the only "fundamental" nonplanar graphs. We then discussed the Four Color Theorem and proved the simpler result that all planar graphs are five-colorable.

53 Exercises

53.1. Give an example of a curve that is closed but not simple.

53.2. Each of the graphs in the figure is planar. Redraw these graphs without crossings.



53.3. Let G be a planar graph with n vertices, m edges, and c components. Let f be the number of faces in a crossing-free embedding of G. Prove that

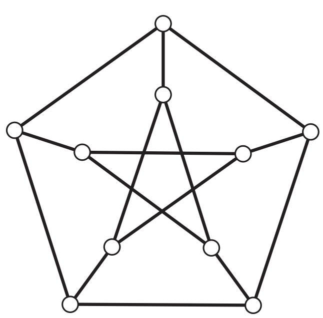
$$n - m + f - c = 1.$$

53.4. Complete the proof of Corollary 53.5. That is, prove that if G is planar, has at least two edges, and does not contain K_3 as a subgraph, then $|E(G)| \le 2|V(G)| - 4$.

53.5. Let G be a graph with 11 vertices. Prove that G or $\overline{G}$ must be nonplanar.

53.6. Let G be a 5-regular graph with ten vertices. Prove that G is nonplanar.

53.7. For which values of n is the n-cube Q_n planar? (See Exercise 52.13.) Prove your answer.



53.8. The graph in the figure is known as *Petersen's graph*. Prove that it is nonplanar by finding either a subdivision of K_5 or a subdivision of $K_{3,3}$ as a subgraph.

53.9. Give a short proof that $\chi(G) \le 6$ for planar graphs (Proposition 53.11) by applying the result of Exercise 52.15 and Corollary 53.6.

53.10. Let $G = (V, E)$ be a planar graph in which every cycle has length 8 or greater.

a. Prove that $|E| \le \frac{4}{3}|V| - \frac{8}{3}$. (You should assume the graph has at least one cycle.)

b. Prove that $\delta(G) \le 2$.

c. Prove that $\chi(G) \le 3$.

53.11. A graph is called *outerplanar* if it can be drawn in the plane so that all the vertices are incident with a common face (which we may take to be the unbounded face). Examples of outerplanar graphs include trees and cycles. Also, if we draw a cycle and add non-crossing diagonal edges, the resulting graph is also outerplanar.

a. Let G be a graph that contains a vertex v that is adjacent to all the other vertices in G. Show that G is planar if and only if $G - v$ is outerplanar.

b. Show that K_4 is not outerplanar.

c. Show that $K_{2,3}$ is not outerplanar.