Graph Theory

Final Project

April 12, 2024

1 Overview

The Final Project will take place over the last two weeks of class; in particular, you will be given the week of April 15 – 19th to research your topic, and you will present your work on the week of April 22 – 26th. Your final project grade will be broken up as follows:

Category	Percentage
Topic Selection	10%
Research Meetings	30%
Presentation	15%
Etiquette	15%
Summary	30%

1.1 Topic Selection

By Monday April 15th you must select your topic, you must email me your selection by 11:59 pm of that day. Also, you must select your presentation date by April 19th, again you must email me your selection by 11:59 pm of that day.

1.2 Research Meetings

You must meet with me at least once during the week of 4/15 – 4/19 and at least once on the week of 4/22 – 4/26, for a total of at least 2 times. These meetings are intended to promote conversations about your research, e..g, what literature you have found, other tools you are using, and any questions you may have. In addition, these meetings serve as a check on what progress you are making. So, if you demonstrate that you have made no progress (due to lack of effort), then you will loose at least 50% of the points for that meeting.

1.3 Presentation

You will present your findings to the class during the week of 4/22 – 4/26. In particular, you will have a 15 minute time slot to present your topic of choice and a summary of what you learned and what progress you made on solving any related open problems.

1.4 Etiquette

You must demonstrate proper etiquette during the final week of presentations. This includes, but is not limited to, attending every class period (even if you are not presenting), being attentive to each presenter, and asking questions. In particular, everyone is responsible for asking at least 2 questions (in total) of other speakers.

1.5 Summary

By 11:59 pm on Friday April 26, you must submit your slides and a three page (single sided, single spaced) summary of your research. Your summary should include a brief introduction of your topic, what results you found in the literature, and what progress you made toward solving any related open problems.

2 Potential Topics

I. The smallest possible fort, or largest possible failed zero forcing set, is of interest. The latter is known as the failed zero forcing number of a graph, see [1, 6, 10]. The former is related to the latter since F is a minimal fort if and only if its complement is a maximal failed zero forcing set [3, Theorem 10]. Moreover, the former is known as the spark of a matrix and has an interesting linear algebra application, see [5, 7]. Below are several open problems:

   a. Identify the failed zero forcing number for families of graphs. The path, cycle, complete, empty, wheel, and certain tree graphs are all known, see [6, Section 3]. What about the hypercube, sunlet, star, spider, or trees in general?

   b. Identify the affect of graph products on the failed zero forcing number. The Cartesian product is considered in [1, 6]. What about joins, corona products, and vertex sums? Consider the results in [3, Section 5].

   c. There is a known construction for forts of a cartesian product, see [4, Proposition 4.1]; however, these forts may not be minimal. Provide a construction for the cartesian product that results in a minimal fort.

   d. Use the spark of a graph to determine lower bounds on the fractional zero forcing number of a graph, see [4].

II. It is known that for graphs of order $n \leq 7$, the maximum nullity is equal to the zero forcing number [2]. The corona product $C_5 \circ K_1$ has zero forcing number 3 and maximum nullity 2. Moreover, one can use vertex sums with copies of this graph to produce a family of graphs for which the maximum nullity and zero forcing number of getting farther apart, see [9, Figure 15]. Below are several open problems:

   a. What other families of graphs can be constructed so that the maximum nullity and zero forcing number get further apart? Note that a catalog of known zero forcing and maximum nullity values can be found at [8]. Further, one can use vertex sums on pendant vertices to easily control the growth of the maximum nullity, see [9, Proposition 3.13].

   b. In [4], it is conjectured that the fort number of a graph is a lower bound on the maximum nullity. Test this conjecture against a family of graphs for which it is known that the maximum nullity and zero forcing number get further apart.

   c. In [4], it is conjectured that the fractional zero forcing number of a graph is a lower bound on the maxim nullity. Test this conjecture against a family of graphs for which it is known that the maximum nullity and zero forcing number get further apart.

III. Recently, the number of minimal forts of a graph was considered [3]. There the authors establish explicit formula and lower bounds for the number of minimal forts of several families of graphs, including the path, cycle, empty, complete, complete bipartite, spider, wheel, and windmill graphs. Below are several open problems:

   a. The windmill graph $Wd(4, k)$ has an exponential number of minimal forts, where the base of the exponent is $3^{1/3}$, which is the largest asymptotic growth rate we have observed, see [3, Corollary 29]. However, we conjecture that there are families of graphs with a larger asymptotic growth rate. Moreover, we believe that the vertex sum can be used to construct such a family.

   b. The number of minimal forts of the spider graph has an asymptotic growth rate that is bounded above by that of a path graph. It is conjectured that the path graph has an extremal growth rate over all trees. Test this conjecture and attempt to prove or disprove it.

c. Dr. Joseph Previte conjectures that there is never less than $n/3$ minimal forts, where $n \geq 1$ is the order of the graph. Test this conjecture and attempt to prove or disprove it.

References

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