Graph Theory

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January 10, 2024

1 Key Topics

Today we introduce fundamental graph definitions and terminology. For further reading, see [1, Section 1.1] and [2, Section 1.1].

1.1 Simple Graphs

Formally, a (simple) graph is a pair of vertices and edges denoted by $G = (V, E)$, where V is a finite set of vertices and E is a set of two element subsets of V known as edges. If unclear from context, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. We often label the vertex set using integers, for example, see the labeled graphs in Figure 1.

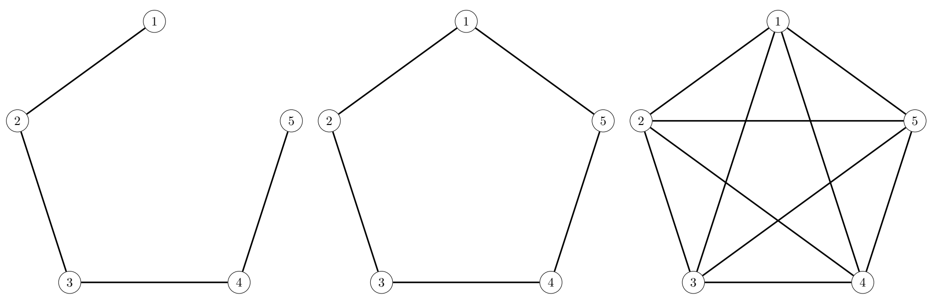

Figure 1: Several labeled graphs on 5 vertices

For simplicity, the edge $\{u, v\}$ can be denoted by uv. Note that $\{u, v\}$ (or uv) is the same as the edge $\{v, u\}$ (or vu). Some authors will also denote the edge $\{u, v\}$ by (u, v), but this should only be used for directed graphs.

A multigraph is a graph that allows for multiple edges between the same pair of vertices. For example, the graph of Konigsberg is a multigraph. For simple graphs G, the edges E are defined as a set of two element subsets of V. Hence, simple graphs do not allow for multiple edges between the same pair of vertices. Similarly, simple graphs do not allow for loops, that is, edges between the same vertex.

1.2 Terminology

We have the following basic graph terminology:

1. The *order* of a graph $G = (V, E)$ is $|V|$,
2. The *size* of a graph $G = (V, E)$ is $|E|$,
3. A graph with no edges is *empty*,
4. A graph with no vertices is *null*,
5. Two vertices $u, v \in V$ are *adjacent* if $\{u, v\} \in E$, we often denote adjacency vertices by $u \sim v$,

6. The *neighborhood* of a vertex $u \in V$ is

$$N(u) = \{v \colon \{u, v\} \in E\},$$

7. The *degree* of a vertex $u \in v$ is $d(u) = |N(u)|$,

8. If $d(u) = 1$, then u is a *pendant vertex*,

9. If $d(u) = 0$, then u is an *isolated* vertex.

Example 1.1. The graph on the left of Figure 1 is an example of a *path graph*; this particular graph has vertices $V = \{1, 2, 3, 4, 5\}$ and edges $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$. Note that $d(1) = 1$ and $d(5) = 1$, while all other vertices have degree 2.

The graph in the middle of Figure 1 is an example of a *cycle graph*; this particular graph has vertices $V = \{1, 2, 3, 4, 5\}$ and edges $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$. Note that every vertex has degree 2.

The graph on the right of Figure 1 is an example of a *complete graph*; this particular graph has vertices $V = \{1, 2, 3, 4, 5\}$ and edges $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$.

2 Exercises

Prove the following results.

a. Let $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$ and $|E| = m$. Then,

$$\sum_{i=1}^{n} d(i) = 2m.$$

b. Every graph has an even number of odd vertices.

c. Let G be a graph with vertex set $V = \{1, 2, 3, 4, 5, 6\}$ and edge set

$$E = \{\{u, v\} \colon u - v = 2k + 1, \; k \in \mathbb{Z}\}.$$

Draw the graph G.

References

[1] D. Joyner, M. V. Nguyen, and D. Phillips, *Algorithmic Graph Theory and Sage*, 2013.

[2] K. Ruohonen, *Graph Theory*, 1st ed., 2013.