Graph Theory

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1 Key Topics

Today we introduce the notion of a subgraph. For further reading, see [1, Section 1.2] and [2, Section 1.1]. Let G and H be graphs. We say that H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Furthermore, we say that H is a *spanning subgraph* if $V(H) = V(G)$. For example, consider the graphs shown in Figure 1: the cycle graph is a spanning subgraph of the complete graph.

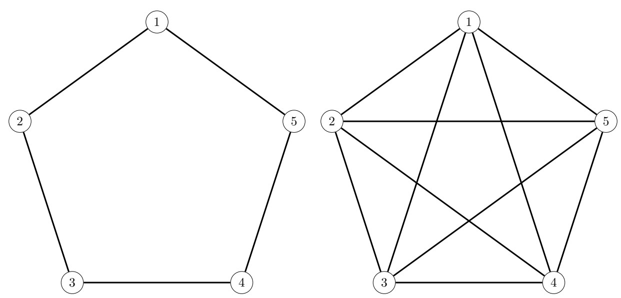

Figure 1: Cycle graph (left) and complete graph (right) of order 5

1.1 Induced Subgraphs

Let $G = (V, E)$ be a graph. The subgraph induced by $E' \subseteq E$ is defined by $H = (V, E')$. Note that the cycle graph in Figure 1 is induced by the complete graph with edge set

$$E' = \{\{1,3\},\{1,4\},\{2,4\},\{2,5\},\{3,5\}\}.$$

The subgraph induced by $V' \subseteq V$ is defined by $H = (V', E')$, where

$$E' = \{\{u,v\} \colon u,v \in V' \ni \{u,v\} \in E\}.$$

For example, consider the graph in Figure 2. The graph induced by $V' = \{1,2,4,5\}$ is the complete graph of order 4.

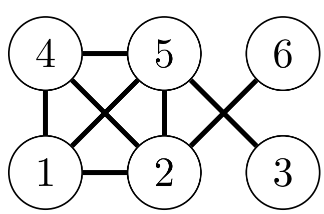

Figure 2: A graph of order 6

Not all subgraphs are induced by a subset of vertices or edges. For example, the graph in Figure 3 is a subgraph of the graph in Figure 2 that is not induced by a subset of vertices or edges.

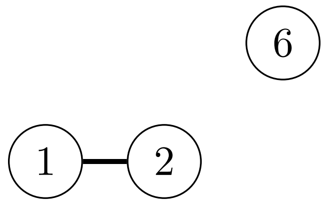

Figure 3: A subgraph not induced by a subset of vertices or edges

1.2 Cliques and Independent Sets

Let $G = (V, E)$ be a graph. A *clique* is a subset of vertices $V' \subseteq V$ such that the induced subgraph is a complete graph. For example, the subset of vertices $V' = \{1, 2, 4, 5\}$ is a clique of the graph in Figure 2. A *maximum clique* is a clique such that there is no clique with more vertices. Moreover, the *clique number* of G, denoted $\omega(G)$, is the cardinality of a maximum clique. The clique number of the graph in Figure 2 is 4.

An *independent set* is a subset of vertices $V' \subseteq V$ such that the induced subgraph is a empty graph. For example, the subset of vertices $V' = \{5, 6\}$ is a independent set of the graph in Figure 2. A *maximum independent set* is a independent set such that there is no independent set with more vertices. Moreover, the *independence number* of G, denoted $\alpha(G)$ is the cardinality of a maximum independent set.

2 Exercises

For each family of graphs, find the clique number and the independence number.

a. The complete graph of order n

b. The cycle graph of order n

c. The star graph of order n

d. The path graph of order n

References

[1] D. Joyner, M. V. Nguyen, and D. Phillips, *Algorithmic Graph Theory and Sage*, 2013.

[2] K. Ruohonen, *Graph Theory*, 1st ed., 2013.