Graph Theory

Thomas R. Cameron

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1 Key Topics

Today, we continue our discussion of subgraphs and introduce the complement graph. For further reading, see [1, Section 1.6.3] and [2, Section 1.3].

2 Cliques and Independent Sets

Let $G = (V, E)$ be a simple graph. Recall that the subgraph induced by the subset $V' \subseteq V$ is defined by $H = (V', E')$, where

$$E' = \{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,5\}\}.$$

Often, we denote the subgraph induced by $V' \subseteq V$ by $G[V']$. A clique is a subset $V' \subseteq V$ such that the induced graph $G[V']$ is a complete graph. An independent set is a subset $V' \subseteq V$ such that the induced graph $G[V']$ is a empty graph. The clique number, denoted $\omega(G)$, is the largest clique in the graph G and the independence number, denoted $\alpha(G)$, is the largest independent set in t he graph G. For example, see the table below.

Graph Family	Clique Number	Independence Number
K_n	$\omega(K_n) = n$	$\alpha(K_n) = 1$
E_n	$\omega(E_n) = 1$	$\alpha(E_n) = n$
C_n	$\omega(C_n) = \begin{cases} 3 & n = 3 \\ 2 & n \neq 3 \end{cases}$	$\alpha(C_n) = \lfloor n/2 \rfloor$
S_n	$\omega(S_n) = 2$	$\alpha(S_n) = n - 1$
P_n	$\omega(P_n) = 2$	$\alpha(P_n) = \lceil n/2 \rceil$

3 Graph Complement

Notice that the clique number and independence number of flipped for the complete and empty graphs. This is not a coincidence since the complete graph and empty graph are complements.

Let $G = (V, E)$ be a simple graph. The *complement* of G, denoted $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and edge set

$$E(\overline{G}) = \{\{u, v\} \colon \{u, v\} \notin E\}.$$

The following proposition states the immediate result that the independence number and clique number are complement parameters.

Proposition 3.1. *Let $G = (V, E)$ be a simple graph. Then, $V' \subseteq V$ is a clique of G if and only if V' is an independent set of $\overline{G}$. Furthermore,*

$$\omega(G) = \alpha(\overline{G}) \text{ and } \alpha(G) = \omega(\overline{G}).$$

The following theorem, originally proved by Frank Ramsey, implies that either the original graph G or its complement must have a "large" clique.

Theorem 3.2. *Let $G = (V, E)$ be a simple graph with $|V| \geq 6$. Then, $\omega(G) \geq 3$ or $\omega(\overline{G}) \geq 3$.*

4 Exercises

I. Prove Theorem 3.2. Hint: Let $v \in V$ and consider two cases: $d(v) \geq 3$ and $d(v) \leq 2$.

II. Review Homework problems 48.10 and 48.11.

References

[1] D. Joyner, M. V. Nguyen, and D. Phillips, *Algorithmic Graph Theory and Sage*, 2013.

[2] K. Ruohonen, *Graph Theory*, 1st ed., 2013.