Graph Theory

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1 Key Topics

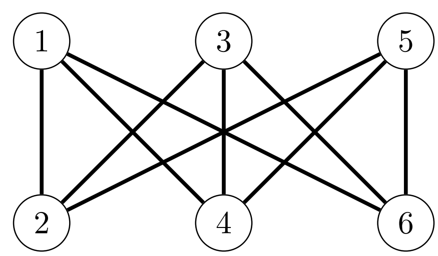
Today, we begin our discussion of what it means for a graph to be connected; in particular, we introduce the concepts of walks, trails, and paths. For further reading, see [1, Section 1.2.1] and [2, Section 1.2].

1.1 Walks

Let $G = (V, E)$. A *walk* is a list of vertices $(v_0, v_1, \ldots, v_l)$ such that $v_i \sim v_{i+1}$ for $i = 0, 1, \ldots, l-1$. The *initial* and *terminal* vertices the walk are v_0 and v_l, respectively. The *length* of the walk is l and the number of vertices in the walk is $l+1$.

Example 1.1. Consider the graph below (right) and each of the walks listed (left).

- $W_1 = (1)$,
- $W_2 = (1, 2)$,
- $W_3 = (2, 1)$,
- $W_4 = (1, 4, 3, 6)$,
- $W_5 = (6, 5, 2, 1)$,
- $W_6 = (1, 4, 3, 2, 1, 4, 5)$,
- $W_7 = (1, 4, 5, 6, 3, 4)$.



Note that every single vertex is a walk of length 0. In addition, two vertices connected by an edge constitutes a walk of length 1. For example, the walk $W_2 = (1, 2)$ is a walk of length 1. Note that W_3 is the reversal of W_2, some authors note this by $W_3 = W_2^{-1}$. □

Consider the two walks

$$W_1 = (v_0, v_1, \ldots, v_l),$$
$$W_2 = (w_0, w_1, \ldots, w_k).$$

If $v_l = w_0$, then the concatenation walk is defined by

$$W_1 + W_2 = (v_0, v_1, \ldots, v_l, w_1, \ldots, w_k).$$

Note that walks W_3 and W_4 from Example 1.1 can be combined as follows:

$$W_4 + W_5 = (1, 4, 3, 6, 5, 2, 1). \tag{1}$$

This is an example of a *closed walk* since it begins and ends at the same vertex.

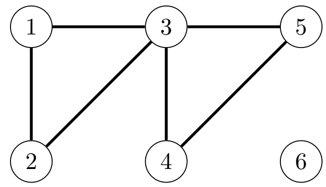


Figure 1: A simple graph with circuit $(1, 2, 3, 5, 4, 3, 1)$

1.2 Trails and Paths

A walk is a *trail* if no edge is used more than once. For instance, in Example 1.1, walk W_5 is a trail while walk W_6 is not. A closed trail (initial and terminal vertices are the same) is called a *circuit*. For example, the walk $W = (1, 2, 3, 5, 4, 3, 1)$ is a circuit in the graph shown in Figure 1

A *trail* is a *path* if no vertex is used more than once, except possibly the initial and terminal vertices. If the path is closed (initial and terminal vertices are the same), then we call the path a *cycle*. For instance, in Example 1.1, walk W_7 is a trail but not a path while walk W_5 is both a path and a trail. Also, note that $W_4 + W_5$ in (1) is a cycle The following proposition states that every path is a trail.

Proposition 1.2. *Let $G = (V, E)$ and let $W = (v_0, v_1, \ldots, v_l)$, where $l \geq 3$, denote a walk of G. If W is a path, then W is a trail.*

Proof. The following is a proof by contrapositive. Suppose that W is not a trail, then there is an edge that gets used more than once. Hence, there exists distinct $i, j \in \{0, 1, \ldots, l-1\}$ such that the edges $e_i = \{v_i, v_{i+1}\}$ and $e_j = \{v_j, v_{j+1}\}$ are the same.

Now, without loss of generality, suppose that $j \geq i + 1$. If $j = i + 1$, then we have the following

$$v_i \sim v_{i+1} = v_j \sim v_{j+1}.$$

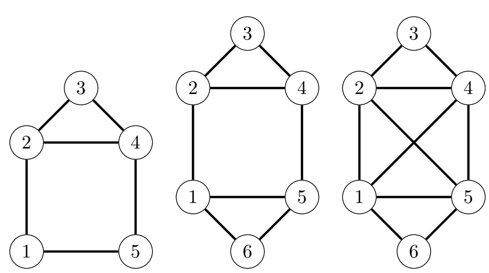
Since the edges e_i and e_j are the same, it follows that $v_i = v_{j+1}$. Furthermore, since the length of W is at least 3, it follows that v_i and v_{j+1} cannot be initial and terminal vertices, respectively, of the walk W. Hence, W is not a path. If $j > i + 1$, then we have the following

$$v_i \sim v_{i+1} \sim \cdots \sim v_j \sim v_{j+1}.$$

Since the edges e_i and e_j are the same, it follows that $v_i = v_j$ $(v_{i+1} = v_{j+1})$ or $v_i = v_{j+1}$ $(v_{i+1} = v_j)$. In either case, W is not a path. $\square$

2 Exercises

An *Eulerian trail* is a trail that visits every edge exactly once. An *Eulerian circuit* is a closed trail that visits every edge exactly once. A *Hamiltonian path* is a path that visits every vertex exactly once. A *Hamiltonian cycle* is a closed path that visits every vertex exactly once. Which of the following graphs have an Eulerian trail/circuit or a Hamiltonian path/cycle?



References

[1] D. Joyner, M. V. Nguyen, and D. Phillips, *Algorithmic Graph Theory and Sage*, 2013.

[2] K. Ruohonen, *Graph Theory*, 1st ed., 2013.