Graph Theory

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1 Key Topics

Today, we continue our discussion of connected graphs and connected components. For further reading, see [1, Section 1.2.1] and [2, Section 1.2].

Recall that a walk is a list of vertices $(v_0, v_1, \ldots, v_l)$ such that $v_i \sim v_{i+1}$ for $i = 0, 1, \ldots, l-1$. We often refer to this list of vertices as a walk from v_0 to v_l. A trail is a walk in which no edge is repeated and a path is a walk in which no vertex is repected, except possibly the initial and terminal vertices. Finally, recall that every path of length $l \geq 3$ is a trail.

1.1 Connected Relation

Let $G = (V, E)$ and let $u, v \in V$. We say that u is *connected to* v if there exists a path from u to v, we reference such a path as a (u,v)-path. In what follows, we show that connected to is an equivalence relation. First, we need the following lemma which states that if there is a walk from u to v, then there exists a path from u to v.

Lemma 1.1. *Let $G = (V, E)$ and let $u, v \in V$. If there exists a (u,v)-walk, then there exists a (u,v)-path.*

Proof. Suppose there exists a (u,v)-walk and define

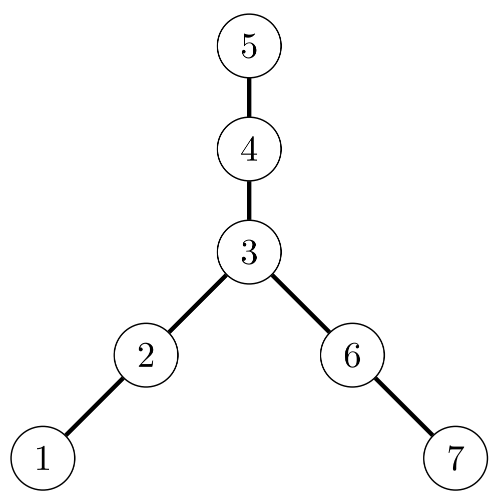
$$S = \{n \in \mathbb{N}\colon \text{there exists a (u,v)-walk of length } n\}.$$

Since $S \subseteq \mathbb{N}$ is non-empty, the well-ordering principal implies that there is a minimum element of S, i.e., there exists an $m \in S$ such that $m \leq k$ for all $k \in S$.

Now, let $W = (v_0, v_1, \ldots, v_m)$ denote a (u,v)-walk of length m. We claim that W is a path. Indeed, if there were a duplicate vertex x in W, then we could remove the vertices between the repeated occurrences of W and attain a (u,v)-walk of length less than m, which contradicts m being the minimum element of S. $\square$

Example 1.2. To see why Lemma 1.1 is necessary in the proof of Theorem 1.2, consider the graph below (right) and the concatenation of two paths (left).

$$\begin{aligned} W_1 &= (1,2,3,4,5) \\ W_2 &= (5,4,3,6,7) \\ W_1 + W_2 &= (1,2,3,4,5,4,3,6,7) \end{aligned}$$



Theorem 1.3. *Let $G = (V, E)$. The connected to relation is an equivalence relation on V.*

Proof. To show that connected to is an equivalence relation, we must show that it is reflexive, symmetric, and transitive. To this end, let $u, v, w \in V$.

For reflexive, note that (u) is a path from u to u; hence, u is connected to itself.

For symmetric, suppose that u is connected to v. Then, there is a path $W = (v_0, v_1, \ldots, v_l)$ such that $u = v_0$ and $v = v_l$. Therefore, the reversal $W^{-1} = (v_l, v_{l-1}, \ldots, v_1, v_0)$ is a path from v to u; hence, v is connected to u.

For transitive, suppose that u is connected to v and v is connected to w. Then, there exists paths $W_1 = (v_0, v_1, \ldots, v_l)$ and $W_2 = (w_0, w_1, \ldots, w_k)$ such that $u = v_0$, $v = v_l = w_0$, and $w = w_k$. Therefore, the concatenation $W = W_1 + W_2$ is a walk from u to w; hence, by Lemma 1.1, there is a path from u to w and it follows that u is connected to w. □

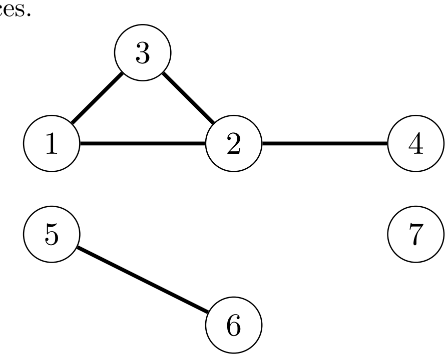
1.2 Connected Components

Every equivalence relation comes with a partition: the equivalence classes of the relation. In the case of the connected to relation, the equivalence classes correspond to disjoint subsets of vertices, where $u, v \in V$ are in the same class if and only if there is a (u,v)-path.

Let $G = (V, E)$ and let $V_1, \ldots, V_k$ denote the equivalence classes under the connected to relation. Then, the *connected components* of G are the induced subgraphs $G[V_i]$, for $i = 1, \ldots, k$.

Example 1.4. Consider the graph below (right) and the adjacency matrix (left) Note how the equivalence classes ($V_1 = \{1, 2, 3, 4\}$, $V_2 = \{5, 6\}$, and $V_3 = \{7\}$) partition the graph into connected components, which can be seen in the adjacency matrix via diagonal block matrices.

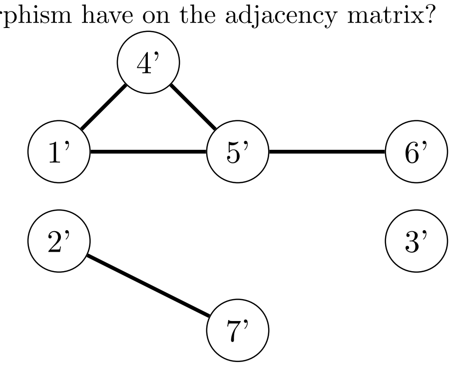
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



2 Exercises

Consider the graph below (right) and the adjacency matrix (left). Find an isomorphism between this graph and the graph in Example 1.4. What impact does this isomorphism have on the adjacency matrix?

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



References

[1] D. Joyner, M. V. Nguyen, and D. Phillips, *Algorithmic Graph Theory and Sage*, 2013.

[2] K. Ruohonen, *Graph Theory*, 1st ed., 2013.