Graph Theory

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1 Key Topics

Today, we introduce the concept of distance and diameter in graphs. For further reading, see [1, Section 5.1]. Recall that a walk from u to v is a list of vertices $(v_0, v_1, \ldots, v_l)$ such that $v_i \sim v_{i+1}$ for $i = 0, 1, \ldots, l-1$ and $u = v_0$ and $v = v_l$. We denote such a walk as a (u,v)-walk. If the walk is a trail, then we reference it as a (u,v)-trail. If the walk is a path, then we reference it as a (u,v)-path.

1.1 Distance

Let $G = (V, E)$ and let $u, v \in V$. The *distance* from u to v is defined by

$$d(u,v) = \begin{cases} \min\{l\colon \text{there is a (u,v)-path of length } l\} & u \text{ is connected to } v \\ \infty & u \text{ is not conntcted to } v \end{cases}$$

The distance function is an example of a metric, similar to the absolute value function, as the following proposition shows.

Proposition 1.1. *Let $G = (V, E)$. Then, the distance function satisfies the following properties:*

a. $d(u,v) \geq 0$ for all $u, v \in V$ and $d(u,v) = 0$ if and only if $u = v$,

b. $d(u,v) = d(v,u)$ for all $u, v \in V$,

c. $d(u,v) \leq d(u,w) + d(w,v)$ for all u, v, w.

Proof. a. It is immediately clear that the distance function is non-negative. Furthermore, $d(u,v) = 0$ if and only if there exists a path of length 0 from u to v, which is true if and only if $u = v$.

b. If u is not connected to v, then $d(u,v) = d(v,u) = \infty$. Now, suppose that u is connected to v and let $W = (v_0, v_1, \ldots, v_m)$ denote a (u,v)-path of minimum length. Then, the reversal W^{-1} is a (v,u)-path of minimum length. Indeed, if there exists a (v,u) path of length less than m, then its reversal will be a (u,v) path of length less than m, which is a contradiction. Hence, $d(u,v) = d(v,u) = m$.

c. Proof saved for Homework 2.

□

1.2 Diameter

Let $G = (V, E)$ denote a connected graph. Given $u \in V$, we define the *eccentricity* of u by

$$\text{ecc}(u) = \max\{d(u,v)\colon v \in V\}.$$

Note that since G is connected the eccentricity of every vertex is finite. Furthermore, the *radius* and *diameter* of a graph are defined as the minimum and maximum eccentricity, respectively:

$$\text{rad}(G) = \min\{\text{ecc}(u) : u \in V\}$$
$$\text{diam}(G) = \max\{\text{ecc}(u) : u \in V\}$$

The *center* of the graph is the set of vertices u such that $\mathrm{ecc}(u) = \mathrm{rad}(G)$; the *periphery* of the graph is the set of vertices u such that $\mathrm{ecc}(u) = \mathrm{diam}(G)$.

For example, the graph in Figure 1 has radius $\mathrm{rad}(G) = 3$, which is attained by vertex 5. In addition, the graph has diameter $\mathrm{diam}(G) = 6$, which is attained by vertices 2 and 8. Therefore, the center of the graph is $\{5\}$ and the periphery of the graph is $\{2, 8\}$.

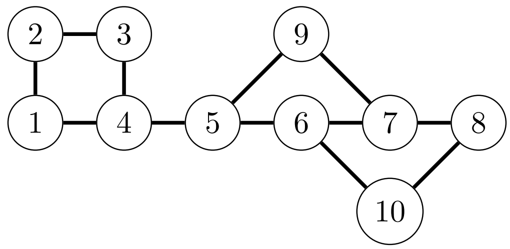

Figure 1: Connected graph with $\mathrm{rad}(G) = 3$ and $\mathrm{diam}(G) = 6$

Theorem 1.2. *Let $G = (V, E)$ be a connected graph. Then,*

$$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G).$$

Proof. By definition, $\mathrm{rad}(G) \leq \mathrm{diam}(G)$, so we only need to prove the upper bound. Let $u, v \in V$ such that $d(u, v) = \mathrm{diam}(G)$ and let c denote a vertex in the center of G. Then,

$$\mathrm{diam}(G) = d(u, v) \leq d(u, c) + d(c, v) \leq 2\,\mathrm{ecc}(c) = 2\,\mathrm{rad}(G).$$

□

Theorem 1.3. *Let $G = (V, E)$. Then, $V(G)$ is the center of some graph.*

Proof. We construct a graph H from G as follows

$$V(H) = V(G) \cup \{w, x, y, z\},$$
$$E(H) = E(G) \cup \{wx, yz\} \cup \{xu\colon u \in V(G)\} \cup \{yu\colon u \in V(G)\}.$$

Now, $\mathrm{ecc}(w) = \mathrm{ecc}(z) = 4$, $\mathrm{ecc}(y) = \mathrm{ecc}(x) = 3$, and for any $v \in V(G)$, $\mathrm{ecc}(v) = 2$. Hence, $V(G)$ is the center of H. □

2 Exercises

Complete the following table

Graph Family	Radius	Diameter
K_n		
C_n		
S_n		
P_n		

References

[1] K. Ruohonen, *Graph Theory*, 1st ed., 2013.