Graph Theory

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1 Key Topics

Today we discuss graph operations, with a particular focus on the Cartesian product. For further reading, see [1, Section 1.6] and [2, Section 1.3].

1.1 Basic Graph Operations

Let $G = (V, E)$ and let $v \in V$ and $e \in E$. Then, the *vertex deletion* of G on v is defined by

$$G - v = (V \setminus \{v\}, E \setminus \{e \in E \colon v \in e\})$$

and the *edge deletion* of G on e is defined by

$$G - e = (V, E \setminus \{e\}).$$

Both of these operations result in a subgraph of G.

Edge contraction is another graph operation that results in a minor of G. Given an edge $e = \{u, v\}$ of G, the *edge contraction* operation first removes e from the edge set and then merges the vertices u and v into a new vertex w, where

$$N(w) = N(u) \setminus \{v\} \cup N(v) \setminus \{u\}.$$

For example, consider the graph and its minor in Figure 1

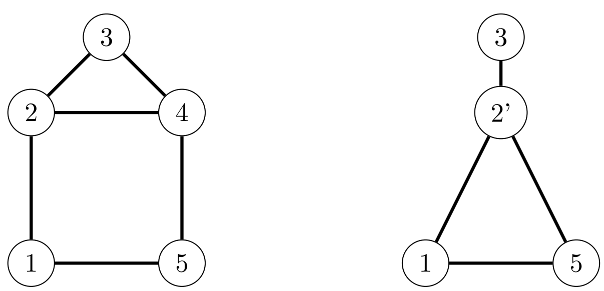


Figure 1: A graph and its minor

Other graph operations, take two graphs to produce a new graph. Let $G = (V, E)$ and $G' = (V', E')$ be simple graphs, where $V \cap V' = \emptyset$. Then, the *disjoint union* of the two graphs is defined by

$$G \cup G' = (V \cup V', E \cup E').$$

Furthermore, the *join* of the two graphs is defined by

$$G \nabla G' = (V \cup V', E \cup E' \cup \{\{v, v'\} \colon v \in V, v' \in V'\}).$$

The disjoint union $P_2 \cup C_4$ and the join $P_2 \nabla C_4$ are shown in Figure 2.

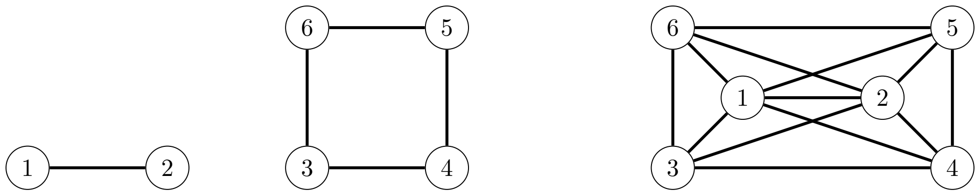


Figure 2: Disjoint union (left) and join (right) of P_2 and C_4

1.2 Cartesian Product

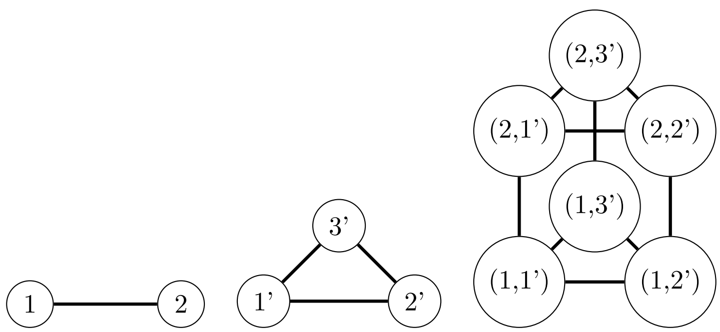
Let $G = (V, E)$ and $G' = (V', E')$ be simple graphs, where $V \cap V' = \emptyset$. Then, the *Cartesian product* of G and G' is denoted by $G \square G'$. The vertex set of $G \square G'$ is given by the Cartesian product:

$$V(G \square G') = V \times V'$$

and the edge set is defined as follows

$$E(G \square G') = \{\{(u, u'), (v, v')\} \colon u = v \text{ and } u' \sim v', \text{ or } u' = v' \text{ and } u \sim v\}$$

For example, the Cartesian product of P_2 and C_3 is shown in Figure



2 Exercises

The d-dimensional hypercube graph is a family of graphs that can be defined recursively:

$$Q_1 = P_2$$
$$Q_d = Q_{d-1} \square P_2, \; d \geq 2.$$

a. Draw the graphs Q_1, Q_2, Q_3, and Q_4.

b. Show that the diameter of Q_d is equal to d.

c. Show that the chromatic number of Q_d is equal to 2.

References

[1] D. Joyner, M. V. Nguyen, and D. Phillips, *Algorithmic Graph Theory and Sage*, 2013.

[2] K. Ruohonen, *Graph Theory*, 1st ed., 2013.