Graph Theory

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1 Key Topics

Last week, we discussed connectivity, distance, and graph products. Today, we continue our discussion of graph products, with a focus on cut vertices and edges. For further reading, see [**?**, Section 1.4].

Recall that a graph is connected if it exactly one connected component, i.e., for every pair of vertices there exists a path between them. So, a graph is *disconnected* if it has more than one connected component, i.e., there exists a pair of vertices with no path between them.

1.1 Cut Vertices and Edges

Let $G = (V, E)$ and let $v \in V$ and $e \in E$. Recall that the vertex of G on v is defined by

$$G - v = (V \setminus \{v\}, E \setminus \{e \in E \colon v \in e\})$$

and the edge deletion of G on e is defined by

$$G - e = (V, E \setminus \{e\}) \,.$$

Now, we say that v is a *cut vertex* if $G - v$ has more connected components than G. Furthermore, we say that e is a *cut edge* if $G - e$ has more connected components than G.

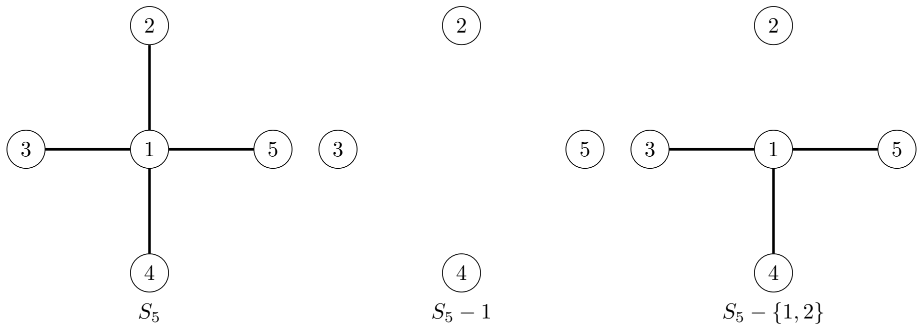

Figure 1: A graph and some of its subgraphs

1.2 Results

Theorem 1.1. *Let $G = (V, E)$ be connected and let $v \in V$. Then, v is a cut vertex if and only if there exists distinct vertices $u, w \in V \setminus \{v\}$ such that v is in every (u,w)-path.*

Proof. Suppose that v is a cut vertex. Then, $G - v$ has at least two connected components: $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Let $u \in V_1$ and $w \in V_2$. Let $W = (v_0, v_1, \ldots, v_l)$ be any (u,w)-path in G; such a path must exist since G is connected. Then, v must be in W; otherwise, W would be a path in $G - v$, which would contradict G_1 and G_2 being distinct connected components.

Conversely, suppose there are distinct vertices $u, w \in V \setminus \{v\}$ such that v is in every (u,w)-path. Then, there is no (u,v)-path in $G - v$; hence, G contains at least two connected components, so v is a cut vertex. $\square$

In a similar manner, we can prove the following result.

Theorem 1.2. *Let $G = (V, E)$ be connected and let $e \in E$. Then, e is a cut edge if and only if there exists distinct vertices $u, w \in V$ such that every (u,w)-path traverses e.*

Theorem 1.3. *Let $G = (V, E)$ be connected and let $e \in E$. If E is a cut edge, then $G - e$ has exactly two connected components.*

2 Exercises

Prove Theorem 1.3 using proof by contradiction.